HECKE ORBITS ON SHIMURA VARIETIES OF HODGE TYPE

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Abstract. We prove the Hecke orbit conjecture of Chai–Oort for Shimura varieties of Hodge type at odd primes of good reduction. We use a novel result for the local monodromy groups of F-isocrystals "coming from geometry", which refines Crew's parabolicity conjecture. In the course of the proof, we also introduce a noncommutative generalisation of Serre–Tate coordinates for formal neighbourhoods of central leaves, built upon the previous work of Caraiani—Scholze and Kim. Using these coordinates, we reinterpret Chai–Oort's notion of strongly Tate-linear subspaces and we establish upper bounds for their monodromy groups. For this step, we employ the notion of Cartier–Witt stacks, as introduced by Drinfeld and Bhatt–Lurie. Another crucial ingredient in the proof is a rigidity result proved by Chai–Oort, which shows that the relevant subspaces are strongly Tate-linear.

Contents

1. Introduction

1.1. **The Hecke orbit conjecture.** Let p be a prime number and g a positive integer. Problem 15 on Oort's 1995 list of open problems in algebraic geometry, [66], is the following conjecture.

Conjecture 1. *Let $x = (A_x, \lambda)$ be an $\overline{\mathbb{F}}_p$-point of the moduli space $\mathcal{A}_g$ of principally polarised abelian varieties of dimension g over $\overline{\mathbb{F}}_p$. The Hecke orbit of x, consisting of all points $y \in \mathcal{A}_g(\overline{\mathbb{F}}_p)$ corresponding to principally polarised abelian varieties related to (A_x, λ) by symplectic isogenies, is Zariski dense in the Newton stratum of $\mathcal{A}_g$ containing x.*

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This article contains a proof of this conjecture. More generally, we prove that for the special fibre of a Shimura variety of Hodge type at an odd prime of good reduction, the isogeny classes are Zariski dense in the Newton strata containing them.

1.1.1. There is a refined version of Conjecture 1, also due to Oort, which considers instead the prime-to-p Hecke orbit of x, consisting of all $y \in \mathcal{A}_g(\overline{\mathbb{F}}_p)$ related to x by prime-to-p symplectic isogenies. In this case, the quasi-polarised p-divisible group $(A_x[p^\infty], \lambda)$ is constant on prime-to-p Hecke orbits (not just constant up to isogeny). Therefore, the prime-to-p Hecke orbit of x is contained in the central leaf

$$C(x) = \left\{ y \in \mathcal{A}_g(\overline{\mathbb{F}}_p) \mid A_y[p^\infty] \simeq_\lambda A_x[p^\infty] \right\},$$

where $\simeq_\lambda$ denotes a symplectic isomorphism. Oort proved in [65] that $C(x)$ is a smooth closed subvariety of the Newton stratum of $\mathcal{A}_g$ containing x. He also conjectured that the prime-to-p Hecke orbit of x is Zariski dense in the central leaf $C(x)$. This conjecture is known as the *Hecke orbit conjecture* (for $\mathcal{A}_g$). Thanks to the Mantovan–Oort product formula, [56, 65], the Hecke orbit conjecture implies Conjecture 1.

Central leaves and prime-to-p Hecke orbits can also be defined for the special fibres of Shimura varieties of Hodge type at primes of good reduction, by work of Hamacher and Kim [37, 47]. The Hecke orbit conjecture for Shimura varieties of Hodge type then predicts that the prime-to-p Hecke orbits of points are Zariski dense in the central leaves containing them (see [55, Question 8.2.1] and [12, Conjecture 3.2]).

The Hecke orbit conjecture naturally splits up into a discrete part and a continuous part. The discrete part states that the prime-to-p Hecke orbit of x intersects each connected component of $C(x)$, and the continuous part states that the Zariski closure of the prime-to-p Hecke orbit of x is equidimensional of the same dimension as $C(x)$. The discrete part of the conjecture is [55, Theorem C] (see [43] for related results). In this paper, we will focus on the continuous part of the conjecture.

1.2. **Main result.** Let (G, X) be a Shimura datum of Hodge type with reflex field E, and assume for simplicity that G^{ad} is $\mathbb{Q}$-simple throughout this introduction. Let $p > 2$ be a prime such that $G = \mathsf{G} \otimes \mathbb{Q}_p$ is quasi-split and split over an unramified extension, let $U_p \subseteq G(\mathbb{Q}_p)$ be a hyperspecial subgroup, and let $U^p \subseteq \mathsf{G}(\mathbb{A}_f^p)$ be a sufficiently small compact open subgroup. Choose a place v of E dividing p and set $E = \mathsf{E}_v$. Let $\mathbf{Sh}_U(\mathsf{G}, \mathsf{X})$ be the canonical model of Shimura variety for (G, X) of level $U := U^p U_p$ over E. Let $\mathscr{S}_U(\mathsf{G}, \mathsf{X})$ be the canonical integral model over $\mathcal{O}_E$ constructed in [52], and let $\mathrm{Sh}_{\mathsf{G},U}$ be its geometric special fibre. Let $C \subseteq \mathrm{Sh}_{\mathsf{G},U}$ be a central leaf as constructed in [37] (cf. [47]).

Theorem I (Theorem 8.3.2)**.** *If $Z \subseteq C$ is a non-empty reduced closed subvariety that is stable under the prime-to-p Hecke operators, then $Z = C$.*

When $\mathrm{Sh}_{\mathsf{G},U}$ is a Siegel modular variety, this result (also for $p = 2$) is due to Chai–Oort, see their forthcoming book [19] for the continuous part and [16] for the discrete part. Their proofs do not generalise to more general Shimura varieties because they rely on the existence of hypersymmetric points in Newton strata, which is usually false for Shimura varieties of Hodge type, see [34]. Moreover, their proof of the continuous part of the conjecture relies on the fact that any point $x \in \mathrm{Sh}_{\mathsf{G},U}(\overline{\mathbb{F}}_p)$ is contained in a large Hilbert modular variety, and they use work of Yu–Chai–Oort, [82], on the Hecke orbit conjecture for Hilbert modular varieties at (possibly ramified) primes. There are many other partial results, e.g. for prime-to-p Hecke orbits of hypersymmetric points in the PEL case, [80], or for prime-to-p Hecke orbits of μ-ordinary points, [9, 42, 59, 72, 84].

We also prove that isogeny classes are dense in the Newton strata containing them, see Theorem 8.4.1. Moreover, we prove results about ℓ-adic Hecke orbits for primes $\ell \neq p$ generalising work of Chai, [11], in the Siegel case, see Theorem 8.6.1.

Remark 1.2.1. The assumption that G^{ad} is $\mathbb{Q}$-simple, can be relaxed at the expense of introducing more notation, see Theorem 8.3.2 for a precise statement. The assumption that $p > 2$ is inherited from the work of Kim [47], and is not necessary for Siegel modular varieties.

Remark 1.2.2. Bragg–Yang prove a potentially good reduction criterion for K3 surfaces, see [6, Theorem 8.10], conditional on the Hecke orbit conjecture for certain orthogonal Shimura varieties, see [*ibid.*, Conjecture 8.2]. In Section 8.5 we explain that our results can be used to prove this conjecture for $p > 2$.

1.3. **Local monodromy of F-isocrystals.** One of the main tools of the proof of Theorem I is the theory of *monodromy groups of F-isocrystals*, defined in [20]. We will prove a new result for the local monodromy groups of F-isocrystals “coming from geometry”, which should be of independent interest. This result is used to prove that the local monodromy groups of the crystalline Dieudonné modules of the universal p-divisible groups over $Z \subseteq C$ (notation as in Theorem I) are “big”. We explain it in a more general setting.

1.3.1. Let X be a smooth connected variety over a perfect field with a closed point $x \in X$ and let $(\mathcal{M}^\dagger, \Phi_{\mathcal{M}^\dagger})$ be a semi-simple overconvergent F-isocrystal over X with constant Newton polygon. Since the Newton polygon is constant, the associated F-isocrystal $(\mathcal{M}, \Phi_{\mathcal{M}})$ admits the slope filtration. The main result of [22] tells us that the *monodromy group* of $\mathcal{M}$ (Definition 2.2.4), denoted by $G(\mathcal{M}, x)$, is the parabolic subgroup $P \subseteq G(\mathcal{M}^\dagger, x)$ associated to the slope filtration. We refine that result for the local monodromy group at x. Let $X^{/x}$ be the formal completion of X in x and write $G(\mathcal{M}^{/x}, x)$ for the monodromy group of the restriction of $\mathcal{M}$ to $X^{/x}$ (see Notation 2.2.5).

Theorem II (Theorem 3.4.4)**.** *The monodromy group $G(\mathcal{M}^{/x}, x)$ of the restriction of $\mathcal{M}$ to $X^{/x}$ is the unipotent radical of the monodromy group $G(\mathcal{M}, x)$.*

When $\mathcal{M}$ is the crystalline Dieudonné module of an ordinary p-divisible group, this result is proved by Chai in [10]. Our proof builds instead on the techniques developed in [22] and uses new descent results for isocrystals from [30, 57]. Since $X^{/x}$ is geometrically simply connected, each isocrystal underlying an isoclinic F-isocrystal over $X^{/x}$ is trivial by [2, Theorem 2.4.1]. This already implies that $G(\mathcal{M}^{/x}, x)$ is unipotent. To relate $G(\mathcal{M}^{/x}, x)$ and the unipotent radical of $G(\mathcal{M}, x)$, we pass through the respective generic points.

Write k for the function field of X and k_x for the function field of $X^{/x}$. We first prove in Theorem 3.2.8 that passing from X to $\operatorname{Spec} k$ we do not change the monodromy group of $\mathcal{M}$, as in the étale setting. Then we show that if we extend k to k^{sep} the monodromy group of $\mathcal{M}$ becomes the unipotent radical of $G(\mathcal{M}, x)$ (Proposition 3.4.2). This means that the extension of scalars from k to k^{sep} kills precisely a Levi subgroup of $G(\mathcal{M}, x)$. Subsequently, using the fact that the field extension $k \subseteq k_x$ is separable, we show that when we extend F-isocrystals from k^{sep} to k_x^{sep}, their slope filtration does not acquire new splittings (Proposition 3.3.5). This is enough to prove that the local monodromy group $G(\mathcal{M}^{/x}, x)$ is the same as the monodromy group over k^{sep}. By the previous part of the argument we then deduce that $G(\mathcal{M}^{/x}, x)$ is precisely the unipotent radical of $G(\mathcal{M}, x)$.

As an additional outcome of our analysis, we prove the following result of independent interest.

Theorem 1.3.2 (Theorem 3.2.4)**.** *If X is an irreducible Noetherian Frobenius-smooth (see Definition 2.2.1) scheme over $\mathbb{F}_p$ with generic point η and $\mathcal{M}$ is an isocrystal over X such that $F^*\mathcal{M} \simeq \mathcal{M}$, then*

$$H^0_{\mathrm{cris}}(\eta, \mathcal{M}_\eta) = H^0_{\mathrm{cris}}(X, \mathcal{M}).$$

With this theorem we are also able to deduce the following consequence.

Corollary 1.3.3 (Corollary 3.2.6). *Let X be a Noetherian Frobenius-smooth scheme over $\mathbb{F}_p$. If $(\mathcal{M}, \Phi_{\mathcal{M}})$ is an F-isocrystal over X with locally constant Newton polygon, then it admits the slope filtration.*

1.4. **An overview of the proof of Theorem I.** The overall structure of the proof of Theorem I is similar to the proof of the ordinary Hecke orbit conjecture in [42] and is based on a strategy implicit in the work of Chai–Oort and sketched to us by Chai in a letter.

To explain the proof we first need to establish some notation. Let $\mathrm{Sh}_{\mathsf{G},U,[b]} \subset \mathrm{Sh}_{\mathsf{G},U}$ be the Newton stratum containing the central leaf C, and let $Z \subset C$ be a reduced closed subvariety that is stable under the prime-to-p Hecke operators, as in the statement of Theorem I. The Newton stratum $\mathrm{Sh}_{\mathsf{G},U,[b]} \subseteq \mathrm{Sh}_{\mathsf{G},U}$ corresponds to an element $[b] \in B(G)$ which has an associated Newton (fractional) cocharacter ν_b. Attached to this cocharacter is a parabolic subgroup P_{ν_b} with unipotent radical U_{ν_b}.

Let Z^{sm} be the smooth locus of Z. Then [42, Corollary 3.3.3] tells us that the monodromy group of the crystalline Dieudonné module $\mathcal{M}$ of the universal p-divisible group over Z^{sm} is isomorphic to P_{ν_b}. Theorem II then tells us that for $x \in Z^{\mathrm{sm}}(\overline{\mathbb{F}}_p)$, the monodromy of $\mathcal{M}$ over $Z^{/x} := \mathrm{Spf}\, \widehat{\mathcal{O}}_{Z,x}$ is equal to U_{ν_b}. We are going to leverage this fact to show that $Z^{/x} = C^{/x}$, which will allow us to conclude that Z^{sm} and hence Z is equidimensional of the same dimension as C.

For this we will construct generalised Serre–Tate coordinates on the formal completion $C^{/x} := \mathrm{Spf}\, \widehat{\mathcal{O}}_{C,x}$. To be precise, we will show that there is a Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra[1] $\mathfrak{a}^+$ governing the structure of $C^{/x}$. For example, if $\mathrm{Sh}_{\mathsf{G},U}$ is a Siegel modular variety and C is the ordinary locus, then $C^{/x}$ is a p-divisible formal group by the classical theory of Serre–Tate coordinates, and $\mathfrak{a}^+ = \mathbb{D}(C^{/x})$ is its Dieudonné module, equipped with the trivial Lie bracket.

More generally the perfection of $C^{/x}$ admits the structure of a (functor to) nilpotent Lie $\mathbb{Q}_p$-algebra(s) whose Dieudonné module is $\mathfrak{a} := \mathfrak{a}^+[\frac{1}{p}]$. More canonically, the perfection of $C^{/x}$ is a trivial torsor for a unipotent formal group $\tilde{\Pi}(\mathfrak{a})$ related to the aforementioned nilpotent Lie algebra by the Baker–Campbell–Hausdorff formula. In the Siegel case, this unipotent formal group is the identity component of the group of self-quasi isogenies (compatible with the polarisation up to a scalar) of the p-divisible group $A_x[p^\infty]$. These results come from the perspective of Caraiani–Scholze, [7], on C and the perspective of Kim, [47], on $C^{/x}$.

Remark 1.4.1. We note that our generalisation of Serre–Tate coordinates is different from the notion of a p-divisible cascade given by Moonen in [63]. Our alternative definition is necessary in our work since in the Hodge type case the deformation spaces we consider generally do not have a cascade structure. Already in the μ-ordinary case, all we can hope for is a *shifted subcascade* in the sense of [73] (cf. [44]) and, as far as we can see, there is no way to run our arguments with shifted subcascades.

[1] See Definition 4.2.1.

Remark 1.4.2. Our notion of generalised Serre–Tate coordinates was discovered independently by Chai–Oort, see [18] and [8]. They use a slightly different perspective and different terminology, see Section 7 for a comparison.

The Dieudonné–Lie $\breve{\mathbb{Q}}_p$-algebra $\mathfrak{a}$ turns out to be isomorphic to $\operatorname{Lie} U_{\nu_b}$ equipped with a natural F-structure coming from $[b]$. If $\mathfrak{b} \subseteq \mathfrak{a} = \operatorname{Lie} U_{\nu_b}$ is an F-stable Lie subalgebra, we construct a formally smooth closed formal subscheme $Z(\mathfrak{b}^+) \subseteq C^{/x}$, which is strongly Tate-linear in the sense of Chai–Oort, see [14]. The construction is such that when $\mathfrak{b} = \mathfrak{a}$, the formal subscheme $Z(\mathfrak{b}^+)$ is $C^{/x}$ itself. It follows from work of Chai–Oort, see [18, Theorem 5.1], that $Z^{/x}$ admits such a description.

Theorem 1.4.3 (Chai–Oort, Theorem 7.1.1)**.** *There is an F-stable Lie subalgebra $\mathfrak{b} \subseteq \mathfrak{a}$ such that $Z^{/x} = Z(\mathfrak{b}^+)$.*

To prove Theorem I, it then suffices to show that the formal subschemes $Z(\mathfrak{b}^+)$ for $\mathfrak{b} \subsetneq \mathfrak{a}$ have small monodromy. In this direction, we prove the following result:

Theorem 1.4.4 (Theorem 6.1.1)**.** *There is a natural closed immersion from the Lie algebra of the monodromy group of $\mathcal{M}$ over $Z(\mathfrak{b}^+)$ to $\mathfrak{b}$.*

By the previous discussion, we know that the Lie algebra of the monodromy group of $\mathcal{M}$ over $Z^{/x}$ is equal to $\mathfrak{a}$. Therefore, if $Z^{/x} = Z(\mathfrak{b}^+)$ for some $\mathfrak{b}$, then $\mathfrak{a} \subseteq \mathfrak{b} \subseteq \mathfrak{a}$, so that $Z^{/x} = Z(\mathfrak{b}^+) = Z(\mathfrak{a}^+) = C^{/x}$.

The proof of Theorem 1.4.4 uses the Cartier–Witt stacks of Drinfeld [29] and Bhatt–Lurie [3]. We use these stacks to "geometrize" the monodromy groups of isocrystals, which makes it possible to give a geometric proof that they are bounded above. The argument with Cartier–Witt stacks happens in Section 6.

Theorem 1.4.3 is related to rigidity results for p-divisible formal groups of Chai, [13], and to rigidity results for biextensions of p-divisible formal groups of Chai–Oort, [17]. We would also like to mention unpublished work of Tao Song which proves rigidity results in the case of p-divisible 4-cascades. As mentioned above, we deduce Theorem 7.1.1 from [18, Theorem 5.1].

Remark 1.4.5. The unipotent formal group $\tilde{\Pi}(\mathfrak{a})$ for $\mathfrak{a} = \operatorname{Lie} U_{\nu_b}$ is closely related to the unipotent group diamond $\tilde{G}_b^{>0}$ introduced in Chapter III.5 of Fargues–Scholze, [33]. To be precise, there should be an isomorphism $\tilde{\Pi}(\mathfrak{a})^\diamond \simeq \tilde{G}_b^{>0}$ of v-sheaves in groups over $\operatorname{Spd} \overline{\mathbb{F}}_p$. In the PEL case, this is explained in [83, proof of Lemma 11.25].

Remark 1.4.6. The first online version of this article contained a proof of Theorem 7.1.1, inspired by [17]; unfortunately, we discovered in 2024 that this proof had a gap. Chai and Oort independently proved the rigidity theorem in [18], which appeared about two months after the first version of our article appeared. We have decided to cite their work instead of trying to fix the gap in our original proof.

1.5. **Structure of the article.** In Section 2 we cover some background theory on p-divisible groups, isocrystals, and prove some technical results about formal schemes. We prove Theorem II in Section 3. In Section 4 we discuss internal-Hom p-divisible groups and groups of (quasi-)automorphisms of p-divisible groups, and introduce Dieudonné–Lie algebras. In Section 5 we discuss central leaves for Shimura varieties of Hodge type and describe their deformation theory in terms of Dieudonné–Lie algebras using work of Kim, [47]. We prove Theorem 1.4.4 in Section 6 and in Section 7 we explain how Theorem 7.1.1 follows from the work of Chai–Oort. In Section 8 we state our main Theorem 8.3.2, a variant for isogeny classes (Theorem 8.4.1) and a variant for ℓ-adic Hecke orbits (Theorem 8.6.1). We also discuss some future directions and applications.

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2. Preliminaries

In this section we will introduce some notation and recall some definitions about Dieudonné modules and isocrystals. First, we prove some preliminary technical result about formal schemes.

2.1. **Formal schemes.** In this article we use the notion of *formal schemes* as defined in [28]. Note that in the affine case, they correspond to *classical affine formal algebraic spaces* in the sense of [76, Tag 0AID]. The admissible topological rings we will mainly consider will be *adic rings* with a finitely generated ideal of definition.

Definition 2.1.1. A *pre-adic* ring* is a topological ring R endowed with the I-adic topology for some finitely generated ideal I. An *adic* ring* is an I-adically complete pre-adic* ring.

If R is an adic* ring over a field κ (not asked to be perfect) with finitely generated ideal of definition I, we write $h_{\mathrm{Spf}\, R} : \mathrm{Alg}_\kappa^{\mathrm{op}} \to \mathrm{Sets}$ for the associated functor of points, where $\mathrm{Alg}_\kappa^{\mathrm{op}}$ is the opposite of the category of κ-algebras endowed with the fpqc topology. This functor of points is defined as

$$\varinjlim_n h_{\mathrm{Spec}\, R/I^n};$$

it is an fpqc sheaf by [76, Tag 0AI2]. If S is another adic* ring, then by [76, Lemma 0AN0] a homomorphism of functors $f : h_{\mathrm{Spf}\, S} \to h_{\mathrm{Spf}\, R}$ comes uniquely from a continuous homomorphism $f : R \to S$. In the text, we implicitly identify $\mathrm{Spf}\, R$ and $h_{\mathrm{Spf}\, R}$.

Lemma 2.1.2. *Let $(A, \mathfrak{m})$ and $(B, \mathfrak{n})$ be complete Noetherian local rings over κ with residue field κ. Let $f : \mathrm{Spf}\, B \to \mathrm{Spf}\, A$ be a monomorphism of affine formal schemes over κ, then it is a closed immersion.*

Proof. We have a local homomorphism of κ-algebras $f : (A, \mathfrak{m}) \to (B, \mathfrak{n})$ that induces an isomorphism on residue fields and such that for every κ-algebra R the induced map $\mathrm{Hom}(B, R) \to \mathrm{Hom}(A, R)$ is injective.

For $j \geq 0$, let $R = B/(\mathfrak{m}B + \mathfrak{n}^j)$ and assume for the sake of contradiction that $\mathfrak{m}B + \mathfrak{n}^j$ is strictly contained in $\mathfrak{n}$. Then the natural map $B \to R$ does not factor through $B/\mathfrak{n}$. Note that there is another natural map $B \to R$ given by

$$B \to B/\mathfrak{n} = A/\mathfrak{m} \to B/\mathfrak{m}B \to R,$$

and these two natural maps are different by our assumption. But they agree after precomposition with A, so this contradicts the injectivity of $\mathrm{Hom}(B, R) \to \mathrm{Hom}(A, R)$, and thus $\mathfrak{m}_B + \mathfrak{n}^j = \mathfrak{n}$.

It follows from [76, Tag 0AMS] then tells us that the closure of the ideal $\mathfrak{m}B$ is equal to $\mathfrak{n}$ and it follows that the closure of the ideal $\mathfrak{m}^j B$ is equal to $\mathfrak{n}^j$. Therefore the map f is taut and thus by [76, Tag 0APU] the topology on B is given by the $\mathfrak{m}$-adic topology. Thus $B/\mathfrak{m}_B^j$ has the discrete topology and it follows that the following diagram is cartesian

$$\begin{array}{ccc} \operatorname{Spec} B/\mathfrak{m}_B^j & \longrightarrow & \operatorname{Spf} B \\ \downarrow & & \downarrow \\ \operatorname{Spec} A/\mathfrak{m}^j & \longrightarrow & \operatorname{Spf} A. \end{array}$$

This tells us that $f : \operatorname{Spf} B \to \operatorname{Spf} A$ is representable, and [76, Tag 0GHZ] allows us to conclude that f is a closed immersion. $\square$

2.2. **Frobenius-smooth schemes and the Tannakian category of isocrystals.** In the study of crystals, there is a special class of schemes with a particularly good behaviour.

Definition 2.2.1. We say that a scheme X over $\mathbb{F}_p$ is *Frobenius-smooth* if the absolute Frobenius $F : X \to X$ is syntomic.

The condition of being Frobenius-smooth is equivalent to X being Zariski locally of the form $\operatorname{Spec} B$ where B has a *finite (absolute) p-basis*, [30, Lemma 2.1.1], which means that there exist elements $x_1, \cdots, x_n$ such that every element $b \in B$ can be uniquely written as

$$b = \sum b_\alpha^p x_1^{\alpha_1} \cdots x_n^{\alpha_n},$$

where $0 \leq \alpha_i \leq p-1$ and $b_\alpha \in B$.

The main examples of Frobenius-smooth schemes that we will encounter are smooth schemes over perfect fields and power series rings over perfect fields. If X is a Noetherian Frobenius-smooth scheme, then it is regular by a result of Kunz (see [76, Tag 0EC0]).

Definition 2.2.2. If X is a scheme over $\mathbb{F}_p$, we denote by $\operatorname{Isoc}(X)$ the category of crystals in *coherent* $\mathcal{O}$-modules on the absolute crystalline site of X with Hom-sets tensored by $\mathbb{Q}$. When X is irreducible, Noetherian, and Frobenius-smooth we define

$$\kappa := \bigcap_{i=1}^{\infty} \Gamma(X, \mathcal{O}_X)^{p^n}$$

to be the *field of constants* of X. Note that κ is a field thanks to [30, Section 3.1.2] and when X is in addition a geometrically connected scheme of finite type over a perfect field, κ coincides with the base field. We also write K for $W(\kappa)[\frac{1}{p}]$.

Proposition 2.2.3 ([30, Corollary 3.3.3])**.** *If X is an irreducible Noetherian Frobenius-smooth scheme, then* $\operatorname{Isoc}(X)$ *is a K-linear Tannakian category.*

This allows us to define the monodromy groups of isocrystals in this situation.

Definition 2.2.4. Let X be an irreducible Noetherian Frobenius-smooth scheme and let $\mathcal{M}$ be an isocrystal over X. We define $\langle \mathcal{M} \rangle$ to be the Tannakian subcategory of $\operatorname{Isoc}(X)$ generated by $\mathcal{M}$. If ξ is an Ω-point for some perfect field Ω, we define $G(\mathcal{M}, \xi)$ to be the Tannaka group of $\langle \mathcal{M} \rangle$ with respect to the fibre functor induced by ξ. We call it the *monodromy group of* $\mathcal{M}$ with respect to ξ.

Notation 2.2.5. In Theorem II, Section 3.4, and Section 6 we also need to consider the monodromy group of the restriction of an isocrystal $\mathcal{M}$, defined on a smooth variety X over a perfect field of characteristic p, to $X^{/x} = \operatorname{Spf} \widehat{\mathcal{O}}_{X,x}$ where $x \in X$ is a closed point. For simplicity in this situation we will rather consider the restriction of $\mathcal{M}$ via the morphism $\operatorname{Spec} \widehat{\mathcal{O}}_{X,x} \to X$, denoted by $\mathcal{M}^{/x}$. Therefore in these sections we will treat $X^{/x}$ as a scheme rather than an affine formal scheme. If ξ is a perfect point of $\operatorname{Spec} \widehat{\mathcal{O}}_{X,x}$ we then define $G(\mathcal{M}^{/x}, \xi)$ as the monodromy group of $\mathcal{M}^{/x}$ over $\operatorname{Spec} \widehat{\mathcal{O}}_{X,x}$ (note that $\operatorname{Spec} \widehat{\mathcal{O}}_{X,x}$ is Frobenius-smooth). Arguing as in [23, Proposition 2.4.8], one can show that this monodromy group is the same as the one constructed considering the restriction to the affine formal scheme $\operatorname{Spf} \widehat{\mathcal{O}}_{X,x}$.

2.3. **p-divisible groups and Dieudonné-theory.** Let R be a semiperfect $\mathbb{F}_p$-algebra and let $A_{\mathrm{cris}}(R)$ be Fontaine's ring of crystalline periods (see [70, Proposition 4.1.3]) with $\varphi : A_{\mathrm{cris}}(R) \to A_{\mathrm{cris}}(R)$ induced by the absolute Frobenius on R. We also denote by $B^+_{\mathrm{cris}}(R)$ the ring $A_{\mathrm{cris}}(R)[\frac{1}{p}]$.

Definition 2.3.1. A *Dieudonné module* over R is a pair (M^+, φ_{M^+}), where M^+ is a finite locally free $A_{\mathrm{cris}}(R)$-module and where

$$\varphi_{M^+} : M^+[\tfrac{1}{p}] \to M^+[\tfrac{1}{p}]$$

is a semilinear bijection such that

$$M^+ \subseteq \varphi_{M^+}(M^+) \subseteq \tfrac{1}{p} M^+.$$

A *rational Dieudonné module* (M, φ_M) over R is a φ-module over $B^+_{\mathrm{cris}}(R)$ which is obtained as $(M^+, \varphi_{M^+})[\frac{1}{p}]$ for a *Dieudonné module* over R is a pair (M^+, φ_{M^+}).

Remark 2.3.2. Note that our definition of Dieudonné modules differs from the original one, where the condition is rather

$$pM^+ \subseteq \varphi_{M^+}(M^+) \subseteq M^+.$$

Our conventions agree instead with the ones in [7]. Thus for a p-divisible group X over R, the covariant Dieudonné module $(\mathbb{D}(X), \varphi_X)$ normalised as in [*ibid.*] is a Dieudonné module over R. In particular, this means that the Dieudonné module of $\mathbb{Q}_p/\mathbb{Z}_p$ over R is $A_{\mathrm{cris}}(R)$ equipped with the standard Frobenius, and the Dieudonné module of μ_{p^∞} is $A_{\mathrm{cris}}(R)$ equipped with the Frobenius divided by p. This also means that the contravariant Dieudonné module is isomorphic to the dual of the covariant Dieudonné module, seen as a F-crystal (see [7, footnote on page 692]). Furthermore, the geometric Frobenius of X is sent by the Dieudonné module functor to the left inverse of φ_X.

Remark 2.3.3. If $R = \overline{\mathbb{F}}_p$, then $A_{\mathrm{cris}}(R) = W(R) = \breve{\mathbb{Z}}_p$. A rational Dieudonné module (M, φ_M) over $\overline{\mathbb{F}}_p$ gives rise to a φ-isocrystal over $\breve{\mathbb{Q}}_p$, whose slopes are contained in $[-1, 0]$. For a p-divisible group X over $\overline{\mathbb{F}}_p$, the slopes are contained in $[0, 1]$ and they are of the form $-\lambda$, where λ is a slope of $\mathbb{D}(X)[\frac{1}{p}]$.

Dieudonné modules are particularly well behaved when R is semiperfect and *quasi-syntomic*.

Definition 2.3.4. We say that a scheme S is *quasi-syntomic* if the relative cotangent complex $L\Omega^1_{S/\mathbb{F}_p}$ has Tor-amplitude in $[-1, 0]$. We also say that $T \to S$ is a *quasi-syntomic cover* if the relative cotangent complex $L\Omega^1_{T/S}$ has Tor-amplitude in $[-1, 0]$ and $T \to S$ is an fpqc cover.

If X and X' are p-divisible groups over a quasisyntomic ring R, then there is a natural map

$$\mathrm{Hom}_R(X, X') \to \mathrm{Hom}_{\varphi, A_{\mathrm{cris}}(R)}(\mathbb{D}(X), \mathbb{D}(X')),$$

where the right hand side denotes homomorphisms of $A_{\mathrm{cris}}(R)$-modules that intertwine φ_X and $\varphi_{X'}$. Now [1, Theorem 4.8.1] tells us that this natural map is an isomorphism if R is quasi-syntomic and semiperfect. Note that the theorem is stated for contravariant Dieudonné-theory with the usual normalisation. The result that we mention follows by taking the dual.

2.3.5. *Representability.* Let us recall some well-known results on the representability of p-divisible groups.

Definition 2.3.6. We say that an affine formal scheme over $\overline{\mathbb{F}}_p$ is a *formal Lie variety* if it is isomorphic to $\mathrm{Spf}\,\overline{\mathbb{F}}_p[[V]]$ for some finite dimensional κ-vector space V. We say that $\dim(V)$ is the dimension of such a formal scheme.

A connected p-divisible group Y over $\overline{\mathbb{F}}_p$, considered as an fppf-sheaf on $\mathrm{Alg}^{\mathrm{op}}_{\overline{\mathbb{F}}_p}$, is representable by a formal Lie variety [60, Chapter II, Theorem 2.1.8] (see also [70, Lemma 3.1.1] for a more general result).

Definition 2.3.7 (See Definition 1.1.4 of [1])**.** We call an $\mathbb{F}_p$-scheme *quasi-regular semiperfect (qrsp)* if it is quasi-syntomic and semiperfect.

2.3.8. By [7, Proposition 4.1.2.(4)], the scheme-theoretic p-adic Tate-module

$$T_pY := \varprojlim_n Y[p^n]$$

is representable by an affine scheme isomorphic to the spectrum of

$$\overline{\mathbb{F}}_p[X_1^{1/p^\infty}, \cdots, X_m^{1/p^\infty}]/(X_1, \cdots, X_m)$$

for some m. This is the classical example of a quasi-regular semiperfect scheme.

We define

$$\overline{\mathbb{F}}_p[[X_1^{1/p^\infty}, \cdots, X_m^{1/p^\infty}]]$$

to be the $(X_1, \cdots, X_m)$-adic completion of $\overline{\mathbb{F}}_p[X_1^{1/p^\infty}, \cdots, X_m^{1/p^\infty}]$. If Y is connected, then the universal cover

$$\tilde{Y} := \varprojlim_{[p]} Y,$$

where the transition maps are given by multiplication by p, is representable by

$$\mathrm{Spf}\,\overline{\mathbb{F}}_p[[X_1^{1/p^\infty}, \cdots, X_m^{1/p^\infty}]]$$

by [70, Proposition 3.1.3.(iii)]. Therefore $\tilde{Y}$ is a filtered colimit of spectra of semiperfect rings, thus it is determined by its values on semiperfect rings. Recall moreover that there is a short exact sequence of fpqc sheaves of abelian groups

$$0 \to T_pY \to \tilde{Y} \to Y \to 0,$$

see [70, Proposition 3.3.1].

2.3.9. Both $\tilde{Y}$ and T_pY are determined by their restriction to the category of semiperfect $\overline{\mathbb{F}}_p$-algebras, and we can describe them explicitly on the category of qrsp $\overline{\mathbb{F}}_p$-algebras as follows.

Lemma 2.3.10. *There is a commutative diagram of natural transformation of functors on the category of qrsp $\overline{\mathbb{F}}_p$-algebras, which evaluated at an object R gives*

$$\begin{array}{ccc} T_pY(R) & \xrightarrow{\simeq} & \Big(A_{\mathrm{cris}}(R) \otimes_{\breve{\mathbb{Z}}_p} \mathbb{D}(Y)\Big)^{\varphi=1} \\ \big\downarrow & & \big\downarrow \\ \tilde{Y}(R) & \xrightarrow{\simeq} & \Big(B^+_{\mathrm{cris}}(R) \otimes_{\breve{\mathbb{Q}}_p} \mathbb{D}(Y)[\tfrac{1}{p}]\Big)^{\varphi=1}, \end{array}$$

where φ is given by the diagonal Frobenius and where $B^+_{\mathrm{cris}}(R) = A_{\mathrm{cris}}(R)[\frac{1}{p}]$.

Proof. Let R be qrsp, then [1, Theorem 4.8.5] tells us that the Dieudonné module functor gives us a natural isomorphism

$$\begin{aligned} T_pY(R) &= \mathrm{Hom}_R((\mathbb{Q}_p/\mathbb{Z}_p)_R, Y_R) \\ &\to \mathrm{Hom}_{A_{\mathrm{cris}},\varphi}(A_{\mathrm{cris}}(R), A_{\mathrm{cris}}(R) \otimes_{\breve{\mathbb{Z}}_p} \mathbb{D}(Y)) \\ &\simeq \Big(A_{\mathrm{cris}}(R) \otimes_{\breve{\mathbb{Z}}_p} \mathbb{D}(Y)\Big)^{\varphi=1} \end{aligned}$$

where the latter bijection is induced by evaluation at 1. Similarly after inverting p we get a natural isomorphism

$$\begin{aligned} \tilde{Y}(R) &= \mathrm{Hom}_R((\mathbb{Q}_p/\mathbb{Z}_p)_R, Y_R)[\tfrac{1}{p}] \\ &\to \Big(B^+_{\mathrm{cris}}(R) \otimes_{\breve{\mathbb{Q}}_p} \mathbb{D}(Y)[\tfrac{1}{p}]\Big)^{\varphi=1} \end{aligned}$$

and the diagram commutes by construction. □

2.3.11. *Complete slope divisibility.* We conclude this section with some recollections on the notion of complete slope divisibility.

Definition 2.3.12. For a perfect ring R we say that an isoclinic Dieudonné module (M^+, φ_{M^+}) over R is *completely slope divisible* if there exist integers s and a with $s \neq 0$ such that $\varphi^s_{M^+} M^+ = p^a M^+$. We also say that a Dieudonné module (M^+, φ_{M^+}) over R is *completely slope divisible* if it is the direct sum of isoclinic completely slope divisible Dieudonné modules, and we say that a p-divisible group is *completely slope divisible* if the associated Dieudonné module is so.

Remark 2.3.13. Since we assumed that R is perfect, the definition we gave is equivalent to the usual one thanks to [67, Proposition 1.3]. Note also that by [*ibid.*, Corollary 1.5], if R is an algebraically closed field, an isoclinic Dieudonné module is completely slope divisible if and only if it is defined over a finite field.

Lemma 2.3.14. *A Dieudonné module over $\overline{\mathbb{F}}_p$ is completely slope divisible if and only if it is a direct sum of isoclinic Dieudonné modules.*

Proof. By [67, Corollary 1.5] it is enough to prove that every isoclinic Dieudonné module is defined over a finite field. By Dieudonné theory, this follows from the fact that a p-divisible group which is geometrically isogenous to a p-divisible group defined over a finite field, is itself defined over a finite field. □

Lemma 2.3.15. *If (M^+, φ_{M^+}) is a completely slope divisible Dieudonné module over $\overline{\mathbb{F}}_p$ and (N^+, φ_{N^+}) is a Dieudonné submodule of (M^+, φ_{M^+}) such that M^+/N^+ is torsion-free, then (N^+, φ_{N^+}) is completely slope divisible.*

Proof. By Lemma 2.3.14, we have to prove that (N^+, φ_{N^+}) is a direct sum of isoclinic Dieudonné modules. By the Dieudonné–Manin classification, there exists a Dieudonné submodule $(\bar{N}^+, \varphi_{\bar{N}^+}) \subseteq (N^+, \varphi_{N^+})$ of finite index which decomposes into a direct sum $\oplus_{\lambda \in \mathbb{Q}} \bar{N}^+_\lambda$ of isoclinic Dieudonné modules. For an element $x \in N^+$, there exist by assumption $x_\lambda \in M^+_\lambda$ for $\lambda \in \mathbb{Q}$ almost all 0 such that $x = \sum_{\lambda \in \mathbb{Q}} x_\lambda$. Since $\bar{N}^+ \subseteq N^+$ is of finite index, there exists n big enough such that $p^n x \in \bar{N}^+$, so that $p^n x_\lambda \in \bar{N}^+$ for every λ. This implies that $p^n x_\lambda \in N^+$ for every λ and by the assumption that M^+/N^+ is torsion-free, we deduce that each x_λ lies in N^+. This yields the desired result. □

3. Local monodromy of F-isocrystals

The main goal of this section is to prove Theorem II. As it often happens, to show the relation between the two Tannakian groups in the statement we first find an equivalent categorical condition. We do this in Section 3.1, where we prove a quite general Tannakian criterion to check that a unipotent subgroup of an algebraic group is the entire unipotent radical.

After that, in Section 3.2, we prove that the global monodromy group of an F-isocrystal with constant Newton polygon is the same as its "generic" monodromy group (Theorem 3.2.8). This will be essential to reduce the entire problem to a problem of F-isocrystals defined over (imperfect) fields with finite p-basis. In Section 3.3 we prove then some descent results, notably the descent of splittings of the slope filtration for separable field extensions with finite p-basis (Proposition 3.3.5), and in Section 3.4 we put all the ingredients together and we prove Theorem II.

3.1. **Two Tannakian criterions.** Let K be a field and let V be a finite-dimensional K-vector space. We recall the following well-known lemma.

Lemma 3.1.1. *If U, U' are unipotent subgroups of $\mathrm{GL}(V)$, then the following two properties are equivalent.*

(i) *$U' \subseteq U$.*

(ii) *$W^U \subseteq W^{U'}$ for every algebraic representation W of $\mathrm{GL}(V)$.*

Proof. The implication (i) ⇒ (ii) is obvious. To prove (ii) ⇒ (i) we just note that by Chevalley's theorem there is a representation W of $\mathrm{GL}(V)$ such that U is the stabiliser of a line $L \subseteq W$. Since U does not admit non-trivial characters, we deduce that $L \subseteq W^U$. By (ii) this implies that U' fixes L and this yields the desired result. □

Thanks to this lemma, we can deduce two Tannakian criterions that we will use later on. For an algebraic group G over a perfect field we will write $R_u(G)$ for the unipotent radical of G. For a cocharacter $\nu : \mathbb{G}_m \to \mathrm{GL}(V)$ we write P_ν for the parabolic subgroup attached to ν and U_ν for $R_u(P_\nu)$.

Proposition 3.1.2. *Suppose K is of characteristic 0 and let $\nu : \mathbb{G}_m \to \mathrm{GL}(V)$ be a cocharacter and $U \subseteq G \subseteq P_\nu$ a chain of subgroups of the parabolic subgroup P_ν. If U and $R_u(G)$ are contained in U_ν, then the following two properties are equivalent.*

(i) *$U = R_u(G)$.*

(ii) *For every representation W of* $\mathrm{GL}(V)$*, the group G stabilises W^U and the induced representation factors through $G/R_u(G)$.*

Proof. The implication (i) $\Rightarrow$ (ii) follows from the observation that $R_u(G)$ is normal in G. For (ii) $\Rightarrow$ (i) first note that by the assumptions we have that $U \subseteq R_u(G)$ since $R_u(G) = G \cap U_\nu$[2]. For the other inclusion, thanks to (ii) we deduce that for every representation W we have that $W^U \subseteq W^{R_u(G)}$. Thus by Lemma 3.1.1, we conclude that $R_u(G) \subseteq U$. □

Proposition 3.1.3. *Let $U \subseteq U'$ be an inclusion of unipotent subgroups of* $\mathrm{GL}(V)$*. The following facts are equivalent.*

(i) $U = U'$.

(ii) $\mathrm{Ext}^1_{U'}(\mathbb{1}, W') \to \mathrm{Ext}^1_U(\mathbb{1}, W'|_U)$ *is injective for every algebraic representation W' of U'.*

Proof. It is clear that (i)$\Rightarrow$ (ii). For the other implication we want to use Lemma 3.1.1. For an algebraic representation W of $\mathrm{GL}(V)$ we consider the *socle filtration* $W_\bullet$ relative to U', namely we define inductively $W_0 = W^{U'}$ and for $i \geq 1$ we define W_i as the preimage of $(W/W_{i-1})^{U'}$ via the quotient $W \to W/W_{i-1}$. The spaces W_i are stabilised by U'. Suppose by contradiction that W^U is not contained in $W^{U'}$. Then there exists a morphism $f\colon \mathbb{1} \to W$ that is U-equivariant, with image contained in some W_i for some $i \geq 1$, and such that the induced morphism $\bar{f}\colon \mathbb{1} \to W_i/W_{i-1}$ is non-zero. This defines a non-trivial extension

$$0 \to W_{i-1} \to E \to \mathbb{1} \to 0$$

of U'-representations which becomes trivial after restriction to U. The existence of such an extension contradicts (ii). This yields the desired result. □

3.2. **Passing to the generic point.** Let X be an irreducible Noetherian Frobenius-smooth scheme over $\mathbb{F}_p$ with generic point η and let $\mathcal{M}$ be an isocrystal over X such that $F^*\mathcal{M} \simeq \mathcal{M}$. In this section we want to prove two results that compare $\mathcal{M}$ with its restriction to the generic fibre $\mathcal{M}_\eta$. For this purpose, we want to use the following theorem by de Jong.

Theorem 3.2.1 ([24, Theorem 1.1])**.** *If $X = \mathrm{Spec}(A)$ is affine with A a DVR and $(\mathcal{M}^+, \Phi_{\mathcal{M}^+})$ is a free F-crystal over X of finite rank, then for every $m, n \in \mathbb{Z} \times \mathbb{Z}_{>0}$ we have*

$$H^0_{\mathrm{cris}}(\eta, \mathcal{M}^+_\eta)^{F^n = p^m} = H^0_{\mathrm{cris}}(X, \mathcal{M}^+)^{F^n = p^m}.$$

Corollary 3.2.2. *If X is as in Theorem 3.2.1 and $\mathcal{M}$ is an isocrystal over X such that $F^*\mathcal{M} \simeq \mathcal{M}$, then*

$$H^0_{\mathrm{cris}}(\eta, \mathcal{M}_\eta) = H^0_{\mathrm{cris}}(X, \mathcal{M}).$$

Proof. Let $(\mathcal{M}^+, \Phi_{\mathcal{M}^+})$ be an F-crystal such that $\mathcal{M}^+[\frac{1}{p}] = \mathcal{M}$. Since X is of dimension 1, by [76, Lemma 0B3N], after possibly replacing $\mathcal{M}^+$ with its double dual we can assume that $\mathcal{M}^+$ is free. We may further assume that the field of constants of A is algebraically closed thanks to [24, §3]. By the Dieudonné–Manin classification both $H^0_{\mathrm{cris}}(\eta, \mathcal{M}_\eta)$ and $H^0_{\mathrm{cris}}(X, \mathcal{M})$ are generated by vectors v such that $\Phi^n_{\mathcal{M}}(v) = p^m v$ for some $m, n \in \mathbb{Z} \times \mathbb{Z}_{>0}$. The result then follows from Theorem 3.2.1. □

[2]Note that unipotent groups U and $G \cap U_\nu$ are connected since we are working in characteristic 0.

We first want to extend de Jong's theorem to more general irreducible Noetherian Frobenius-smooth schemes by using a Hartogs' argument. Thanks to [2, Section 1.1], a ring B with a p-basis admits a p-adic lift $\tilde{B} \twoheadrightarrow B$. By [textitibid., Proposition 1.3.3], the datum of a crystal in quasi-coherent $\mathcal{O}$-modules over $\operatorname{Spec}(B)$ is then equivalent to the datum of a completed $\tilde{B}$-module M^+ and topologically p-nilpotent derivations of M^+ associated to some choice of a lift of a p-basis of B to $\tilde{B}$. We need the following lemma.

Lemma 3.2.3. *Let $f\colon A \to A'$ be an injective morphism of Noetherian Frobenius-smooth rings which sends p-bases to p-bases, and write $\tilde{f}\colon \tilde{A} \to \tilde{A}'$ for a p-adic lift of f. For an isocrystal $\mathcal{M}$ over* $\operatorname{Spec}(A)$ *write $\mathcal{M}'$ for the pullback to* $\operatorname{Spec}(A')$ *and M, M' for the associated modules over $\tilde{A}[\frac{1}{p}]$ and $\tilde{A}'[\frac{1}{p}]$. The following diagram is cartesian*

$$\begin{array}{ccc} H^0_{\mathrm{cris}}(\operatorname{Spec}(A), \mathcal{M}) & \longrightarrow & M \\ \downarrow & & \downarrow \\ H^0_{\mathrm{cris}}(\operatorname{Spec}(A'), \mathcal{M}') & \longrightarrow & M'. \end{array}$$

Proof. By [30, Proposition 3.5.2], the module M is projective, which implies that $M \to M'$ is injective. We choose a lift $\{\tilde{x}_1, \dots, \tilde{x}_n\} \subseteq \tilde{A}$ of a p-basis of A. By the assumption, this is sent by $\tilde{f}$ to a lift of a p-basis of A'. This choice then defines differential operators $\partial_1, \dots, \partial_n$ of M' that stabilise $M \subseteq M'$. By [2, Proposition 1.3.3], this implies that $H^0_{\mathrm{cris}}(\operatorname{Spec}(A'), \mathcal{M}') \subseteq M'$ (resp. $H^0_{\mathrm{cris}}(\operatorname{Spec}(A), \mathcal{M}) \subseteq M$) is the subspace of elements killed by $\partial_1, \dots, \partial_n$. This ends the proof. □

Theorem 3.2.4. *If X is an irreducible Noetherian Frobenius-smooth scheme over $\mathbb{F}_p$ and $\mathcal{N}$ is a subquotient of an isocrystal $\mathcal{M}$ over X such that $F^*\mathcal{M} \simeq \mathcal{M}$, then*

$$H^0_{\mathrm{cris}}(\eta, \mathcal{N}_\eta) = H^0_{\mathrm{cris}}(X, \mathcal{N}).$$

Proof. By [32, Theorem 5.10], we know that $\mathcal{N}$ is a subobject of some isocrystal $\mathcal{M}'$ such that $F^*\mathcal{M}' \simeq \mathcal{M}'$. Thanks to [*ibid.*, Lemma 5.6], it is then enough to prove the result for an isocrystal $\mathcal{M}$ such that $F^*\mathcal{M} \simeq \mathcal{M}$. By Zariski descent, we may further assume that $X = \operatorname{Spec}(A)$ is affine.

Let $\tilde{A}$ a p-adic lift of A and let $\tilde{A}_\eta$ be a p-adic lift of $\operatorname{Frac}(A)$ equipped with a morphism $\tilde{A} \hookrightarrow \tilde{A}_\eta$ lifting the inclusion $A \subseteq \operatorname{Frac}(A)$. We write S for the set of prime ideals $\mathfrak{p}$ of $\tilde{A}$ of codimension 1 containing p and for $\mathfrak{p} \in S$ we write $\tilde{A}_\mathfrak{p} \subseteq \tilde{A}_\eta$ for the p-adic completion of the localisation of $\tilde{A}$ at $\mathfrak{p}$. By construction, we have that $\tilde{A}_\mathfrak{p}/p = A_\mathfrak{p}$. This implies that $\bigcup_{\mathfrak{p} \in S} \tilde{A}_\mathfrak{p} \subseteq \tilde{A}_\eta$ is dense with respect to the p-adic topology since $\bigcup_{\mathfrak{p} \in S} A_\mathfrak{p} = \operatorname{Frac}(A)$.

We first want to prove that the ring

$$\tilde{B} \coloneqq \bigcap_{\mathfrak{p} \in S} \tilde{A}_\mathfrak{p}$$

is equal to $\tilde{A}$. To do this, we first note that $\tilde{B}$ is p-adically complete and p-torsion-free since each $\tilde{A}_\mathfrak{p}$ is so. In addition, by the p-torsion-freeness, we have that $p\tilde{B} = \bigcap_{\mathfrak{p} \in S} p\tilde{A}_\mathfrak{p}$, which implies that the morphism

$$\tilde{B}/p \to \bigcap_{\mathfrak{p} \in S} \tilde{A}_\mathfrak{p}/p \subseteq \tilde{A}_\eta/p$$

is injective. On the other hand, thanks to the algebraic Hartogs' lemma, we have that $\tilde{A}/p = \bigcap_{\mathfrak{p} \in S} \tilde{A}_\mathfrak{p}/p$. This implies that $\tilde{A}/p = \tilde{B}/p$ and in turn this shows that $\tilde{A} = \tilde{B}$.

Now, write M for the module over $\tilde{A}[\frac{1}{p}]$ associated to $\mathcal{M}$ and for every $\mathfrak{p} \in S$ write $M_{\mathfrak{p}}$ for the extension of scalars to $\tilde{A}_{\mathfrak{p}}[\frac{1}{p}]$. By [30, Proposition 3.5.2], we have that M is a direct summand of $\tilde{A}[\frac{1}{p}]^{\oplus n}$ for some $n > 0$. Combining this with the fact that $\tilde{A}[\frac{1}{p}] = \bigcap_{\mathfrak{p} \in S} \tilde{A}_{\mathfrak{p}}[\frac{1}{p}]$, we deduce that

$$M = \bigcap_{\mathfrak{p} \in S} M_{\mathfrak{p}}. \tag{3.2.1}$$

By Lemma 3.2.3 applied to the inclusion $A \subseteq \mathrm{Frac}(A)$, if we denote by M_η the $\tilde{A}_\eta[\frac{1}{p}]$-module associated to $\mathcal{M}_\eta$, we get the cartesian square

$$\begin{array}{ccc} H^0_{\mathrm{cris}}(\mathrm{Spec}(A), \mathcal{M}) & \longrightarrow & M \\ \downarrow & \square & \downarrow \\ H^0_{\mathrm{cris}}(\eta, \mathcal{M}_\eta) & \longrightarrow & M_\eta. \end{array}$$

It remains to prove that every section $v \in H^0_{\mathrm{cris}}(\eta, \mathcal{M}_\eta)$ is also contained in M. By (3.2.1), this is equivalent to showing that v is in $M_{\mathfrak{p}}$ for every $\mathfrak{p} \in S$, which follows from Corollary 3.2.2. $\square$

Remark 3.2.5. Theorem 3.2.4 improves [31, Theorem 2.2.3]. As far as we can see, even if X is a connected smooth variety over a perfect field, the result in [*ibid.*] is not enough to deduce the full faithfulness of the restriction functor to the generic point.

Theorem 3.2.4 has as a corollary the following result that is not essential in our article, but it is of great interest.

Corollary 3.2.6. *Let X be a Noetherian Frobenius-smooth scheme over $\mathbb{F}_p$. If $(\mathcal{M}, \Phi_{\mathcal{M}})$ is an F-isocrystal over X with locally constant Newton polygon, then it admits the slope filtration.*

Proof. First note that we may assume that X is irreducible. As in [45], by taking exterior powers, it is enough to prove that if $(\mathcal{M}, \Phi_{\mathcal{M}})$ has minimal slope of multiplicity 1, then there exists a rank 1 sub-F-isocrystal of $(\mathcal{M}, \Phi_{\mathcal{M}})$ of minimal slope. Note also that the result is known on the generic point η of X (see [*ibid.*] and [25, Claim 2.8]). If $S_1(\mathcal{M}_\eta) \subseteq \mathcal{M}_\eta$ is the subobject of minimal slope, up to taking a power of the Frobenius structure for some $s > 0$ and a Tate twist, we may assume that it corresponds to a lisse $\mathbb{Q}_{q^s}$-sheaf $\mathcal{F}_\eta$ over η. This lisse sheaf admits models over every codimension 1 point by [25, Proposition 2.10], thus it admits an extension to a lisse $\mathbb{Q}_{q^s}$-sheaf $\mathcal{F}$ over X by Zariski–Nagata purity theorem. The lisse sheaf $\mathcal{F}$ corresponds then to an F^s-isocrystal $(\mathcal{N}, \Phi_{\mathcal{N}})$ over X which, by Theorem 3.2.4, embeds in $(\mathcal{M}, \Phi^s_{\mathcal{M}})$ providing a model of the inclusion $S_1(\mathcal{M}_\eta) \subseteq \mathcal{M}_\eta$. This yields the desired result. $\square$

Remark 3.2.7. Note that in [46] Kedlaya proves the analogue of Corollary 3.2.6 for perfect schemes using arc-descent.

We also prove a stronger form of Theorem 3.2.4 under the additional assumption that $\mathcal{M}$ upgrades to an F-isocrystal with slope filtration.

Theorem 3.2.8. *Let X be an irreducible Noetherian Frobenius-smooth scheme over $\mathbb{F}_p$. If $(\mathcal{M}, \Phi_{\mathcal{M}})$ is an F-isocrystal over X with slope filtration, then*

$$G(\mathcal{M}, \eta^{\mathrm{perf}}) = G(\mathcal{M}_\eta, \eta^{\mathrm{perf}}).$$

Proof. Let K be the fraction field of $W(\kappa)$ with κ the field of constants of X and let K' be the fraction field of the ring of Witt vectors of η^{perf}. Thanks to [77, Proposition 3.1.8] applied with $F = K, F' = K$, and $F'' = K'$, it is enough to prove that $\langle \mathcal{M} \rangle \to \langle \mathcal{M}_\eta \rangle$ is fully faithful and sends semi-simple objects to semi-simple objects. The first part is proven in Theorem 3.2.4 and does not need the assumption on the slope filtration. For the second part, for an irreducible object $\mathcal{N} \in \langle \mathcal{M} \rangle$, we want to prove that $\mathcal{N}_\eta$ is semi-simple.

Since $\mathcal{N}$ is irreducible, it is a subquotient of $\mathcal{M}^{\otimes m} \otimes (\mathcal{M}^\vee)^{\otimes n}$ for some $m, n \geq 0$ and by the assumption $\mathcal{M}^{\otimes m} \otimes (\mathcal{M}^\vee)^{\otimes n}$ can be endowed with a Frobenius structure with slope filtration. After taking the sth-power of the Frobenius structure for some $s > 0$ and making a Tate twist, we may further assume that $\mathcal{N}$ appears in the unit-root part of an F^s-isocrystal. Therefore, taking a Jordan–Hölder filtration, we may assume that $\mathcal{N}$ is a subquotient of an isocrystal $\mathcal{M}'$ which admits a unit-root F^s-structure $\Phi_{\mathcal{M}'}$ such that $(\mathcal{M}', \Phi_{\mathcal{M}'})$ is semi-simple. By [2, Theorem 2.4.1], the F^s-isocrystal $(\mathcal{M}', \Phi_{\mathcal{M}'})$ corresponds to a semi-simple lisse $\mathbb{Q}_{p^s}$-sheaf over X. By the regularity of X, the lisse sheaf remains semi-simple when restricted to the generic point. This implies that $(\mathcal{M}'_\eta, \Phi_{\mathcal{M}'_\eta})$ is semi-simple. To conclude we have to prove that $\mathcal{M}'_\eta$ is semi-simple as well. Let $\mathcal{N}'_\eta \subseteq \mathcal{M}'_\eta$ be the *socle* of $\mathcal{M}'_\eta$, namely the sum of all the irreducible subobjects of $\mathcal{M}'_\eta$. By maximality, $\mathcal{N}'_\eta$ is stabilised by the F^s-structure, thus it upgrades to a subobject $(\mathcal{N}'_\eta, \Phi_{\mathcal{N}'_\eta}) \subseteq (\mathcal{M}'_\eta, \Phi_{\mathcal{M}'_\eta})$. By semi-simplicity, the inclusion admits a retraction, which induces in particular a retraction of $\mathcal{N}'_\eta \subseteq \mathcal{M}'_\eta$. This implies that $\mathcal{N}'_\eta = \mathcal{M}'_\eta$, as we wanted. □

3.3. **Descent for isocrystals.** We prove now various descent results that we will need in the next section for (F-)isocrystals. Let $f : Y \to X$ be a pro-étale Π-cover of Noetherian Frobenius-smooth schemes over $\mathbb{F}_p$ where Π is a profinite group and let $y \in Y(\Omega)$ be an Ω-point of Y with Ω a perfect field.

Lemma 3.3.1. *For every isocrystal $\mathcal{M}$ over X, the maximal trivial subobject of $f^*\mathcal{M}$ descends to a subobject $\mathcal{N} \subseteq \mathcal{M}$. Moreover, if $\mathcal{M}$ is endowed with a Frobenius structure $\Phi_{\mathcal{M}}$, the inclusion $\mathcal{N} \subseteq \mathcal{M}$ upgrades to an inclusion $(\mathcal{N}, \Phi_{\mathcal{N}}) \subseteq (\mathcal{M}, \Phi_{\mathcal{M}})$ of F-isocrystals and $(\mathcal{N}, \Phi_{\mathcal{N}})$ is a direct sum of isoclinic F-isocrystals.*

Proof. Since the cover $Y \to X$ is a quasi-syntomic cover, it satisfies descent for isocrystals thanks to [30, Proposition 3.5.4] (see also [57] or [4, Section 2]). By the assumption,

$$Y \times_X Y \simeq \varprojlim_{U \subseteq \Pi} (Y \times_X Y)^U$$

where the limit runs over all the open normal subgroups of Π and $(Y \times_X Y)^U := \coprod_{[\gamma] \in \Pi/U} Y_{[\gamma]}$ is a disjoint union of copies of Y. The group Π acts on $Y \times_X Y$ in the obvious way. Since $f^*\mathcal{M}$ comes from X, it is endowed with a descent datum with respect to the cover $Y \to X$. This datum consists of isomorphisms $\gamma^* \mathcal{M}_{(Y \times_X Y)^U} \simeq \mathcal{M}_{(Y \times_X Y)^U}$ for each $U \subseteq \Pi$ and $\gamma \in \Pi$. The functor γ^* sends trivial objects to trivial objects, which implies that the descent datum restricts to a descent datum of $\mathcal{T}$, the maximal trivial subobject of $f^*\mathcal{M}$. Therefore, $\mathcal{T}$ descends to a subobject $\mathcal{N} \subseteq \mathcal{M}$, as we wanted. If $\mathcal{M}$ is endowed with a Frobenius structure, then it induces a Frobenius structure on each isocrystal $\mathcal{M}_{(Y \times_X Y)^U}$ and this structure preserves each maximal trivial subobject of given slope. This implies that the descended object $\mathcal{N} \subseteq \mathcal{M}$ is stabilised as well by the Frobenius and the induced Frobenius structure satisfies the desired property. □

Proposition 3.3.2. *Let $(\mathcal{M}, \Phi_{\mathcal{M}})$ be an F-isocrystal with the slope filtration and write ν for the associated Newton cocharacter. If $R_u(G(\mathcal{M}, f(y))) \subseteq U_\nu$ and $\mathrm{Gr}_{S_\bullet}(f^*\mathcal{M})$ is trivial, then $G(f^*\mathcal{M}, y) = R_u(G(\mathcal{M}, f(y)))$.*

Proof. Since $\mathrm{Gr}_{S_\bullet}(f^*\mathcal{M})$ is trivial, the group $G(f^*\mathcal{M}, y)$ is a unipotent subgroup of $G(\mathcal{M}, f(y))$ sitting inside U_ν. Therefore, we are in the situation of Proposition 3.1.2 and we have to prove that (ii) is satisfied. This amounts to show that for every $m, n \geq 0$, the maximal trivial subobject $\mathcal{T} \subseteq f^*(\mathcal{M}^{\otimes m} \otimes (\mathcal{M}^\vee)^{\otimes n})$ descends to a semi-simple isocrystal over X. By Lemma 3.3.1, we know that $\mathcal{T}$ descends to an isocrystal $\mathcal{N}$ which is the direct sum of isocrystals which can be endowed with an isoclinic Frobenius structure. Since $R_u(G(\mathcal{M}, f(y)))$ is contained in U_ν, we deduce that $\mathcal{N}$ is semi-simple, as we wanted. □

Lemma 3.3.3. *If k'/k is a separable field extension and k' admits a finite p-basis, then $k' \otimes_k k'$ admits a finite p-basis as well.*

Proof. Thanks to [58, Theorem 26.6], the field k admits a finite p-basis $t_1, \cdots, t_d$ which extends to a finite p-basis $t_1, \cdots, t_d, u_1, \cdots u_e$ of k'. We claim that $\Gamma \coloneqq \{t_i \otimes 1\}_i \cup \{u_i \otimes 1\}_i \cup \{1 \otimes u_i\}_i$ is a finite p-basis of $k' \otimes_k k'$. It is clear from the construction that the elements of Γ generate $k' \otimes_k k'$ over $(k' \otimes_k k')^p$. On the other hand, the exact sequence

$$0 \to \Omega^1_{k/\mathbb{F}_p} \otimes_k (k' \otimes_k k') \to \Omega^1_{k' \otimes_k k'/\mathbb{F}_p} \to (\Omega^1_{k'/k} \otimes_k k') \oplus (k' \otimes_k \Omega^1_{k'/k}) \to 0$$

shows that the elements $d\gamma$ with $\gamma \in \Gamma$ form a basis of the free module $\Omega^1_{k' \otimes_k k'/\mathbb{F}_p}$. We deduce the p-independence of the elements of Γ by arguing as in [76, Lemma 07P2]. □

Lemma 3.3.4. *Let X be a Frobenius-smooth scheme over $\mathbb{F}_p$ and $(\mathcal{M}, \Phi_{\mathcal{M}})$ an F-isocrystal over X with a locally-free lattice and constant Newton polygon. If all the slopes of $(\mathcal{M}, \Phi_{\mathcal{M}})$ are non-zero, the vector space $H^0_{\mathrm{cris}}(X, \mathcal{M})^{F=1}$ vanishes.*

Proof. Since X is Frobenius-smooth, by [2, §1.3.5.ii] the global sections of any isocrystal over X embed into the global sections of the base change to X^{perf}. Over X^{perf} we argue as in the proof of [45, Theorem 2.4.2], namely we assume that $X^{\mathrm{perf}} = \mathrm{Spec}(A)$ is affine and we embed A into a product of perfect fields. This reduces the problem to the case of perfect fields, where the result is well-known. □

Proposition 3.3.5. *Let $k \subseteq k'$ be a separable extension of characteristic p fields with finite p-basis and let $(\mathcal{M}, \Phi_{\mathcal{M}})$ be a free F-isocrystal over k with slope filtration $S_\bullet$ of length n. If $\mathcal{M}_{k'}$ admits a Frobenius-stable splitting of the form $\mathcal{N}_{k'} \oplus S_{n-1}(\mathcal{M}_{k'})$ with $\mathcal{N}_{k'}$ some subobject of $\mathcal{M}_{k'}$, the same is true for $\mathcal{M}$.*

Proof. Since $\mathrm{Spec}\, k' \to \mathrm{Spec}\, k$ is a quasi-syntomic cover, it satisfies descent for isocrystals thanks to the descent results of Drinfeld and Mathew in [30, 57] (see [4, Theorem 2.2]). Therefore, in order to descend $\mathcal{N}_{k'}$ to k it is enough to show that the splitting $\mathcal{N}_{k' \otimes_k k'} \oplus S_{n-1}(\mathcal{M}_{k' \otimes_k k'})$ is unique. Suppose that $\mathcal{N}'_{k' \otimes_k k'} \oplus S_{n-1}(\mathcal{M}_{k' \otimes_k k'})$ was a different splitting. Then there would exist a non-trivial Frobenius-equivariant morphism $\mathcal{N}'_{k' \otimes_k k'} \to S_{n-1}(\mathcal{M}_{k' \otimes_k k'})$. In other words, the F-isocrystal $\underline{\mathrm{Hom}}(\mathcal{N}'_{k' \otimes_k k'}, S_{n-1}(\mathcal{M}_{k' \otimes_k k'}))$ would have a non-trivial Frobenius-invariant global section. Since the slopes of $\underline{\mathrm{Hom}}(\mathcal{N}'_{k' \otimes_k k'}, S_{n-1}(\mathcal{M}_{k' \otimes_k k'}))$ are all negative by definition and $k' \otimes_k k'$ admits a finite p-basis by Lemma 3.3.3, this would contradict Lemma 3.3.4. □

Lemma 3.3.6. *If $f : Y \to X$ is a finite flat surjective morphism of Noetherian Frobenius-smooth schemes and $\mathcal{M}$ is an isocrystal over X, then the natural morphism $H^1_{\mathrm{cris}}(X, \mathcal{M}) \to H^1_{\mathrm{cris}}(Y, f^*\mathcal{M})$ is injective for every i.*

Proof. Thanks to the Leray spectral sequence, we have a natural injective morphism

$$H^1_{\text{cris}}(X, f_*f^*\mathcal{M}) \hookrightarrow H^1_{\text{cris}}(Y, f^*\mathcal{M}).$$

Since a composition of injective morphisms is injective, it remains to prove that the natural morphism $H^1_{\text{cris}}(X, \mathcal{M}) \to H^1_{\text{cris}}(X, f_*f^*\mathcal{M})$ given by the adjunction is injective. For this we note that thanks to [30, Proposition 3.5.2] locally on X the isocrystal $\mathcal{M}$ is a vector bundle over a p-adic lift of X endowed with a p-adic connection. Thus by the projection formula, we have that $f_*f^*\mathcal{M} = \mathcal{M} \otimes_{\mathcal{O}} f_*\mathcal{O}$. We deduce that there is a canonical trace morphism $f_*f^*\mathcal{M} \to \mathcal{M}$ such that the composition

$$\alpha \colon \mathcal{M} \to f_*f^*\mathcal{M} \to \mathcal{M}$$

is the endomorphism of $\mathcal{M}$ that, locally on X, is given by the multiplication with respect to the degree of f. In particular, α is an automorphism of $\mathcal{M}$, so that $H^1_{\text{cris}}(X, \mathcal{M}) \to H^1_{\text{cris}}(X, f_*f^*\mathcal{M})$ admits a retraction. □

Proposition 3.3.7. *If $f \colon Y \to X$ is a finite flat surjective morphism of irreducible Noetherian Frobenius-smooth schemes and $\mathcal{M}$ is an isocrystal over X with unipotent monodromy group, then*

$$G(f^*\mathcal{M}, y) = G(\mathcal{M}, f(y)).$$

Proof. We want to use the criterion of Proposition 3.1.3 applied to the inclusion $G(f^*\mathcal{M}, y) \subseteq G(\mathcal{M}, f(y))$. Thanks to Lemma 3.3.6, we have that $\text{Ext}^1_{\text{Isoc}(X)}(\mathcal{O}_X, \mathcal{N}) \to \text{Ext}^1_{\text{Isoc}(Y)}(\mathcal{O}_Y, f^*\mathcal{N})$ is injective for every isocrystal $\mathcal{N}$ over X. It remains to note that if κ is the field of constants of Y, then the extension of scalars from $W(\kappa)[\frac{1}{p}] \subseteq W(\Omega)[\frac{1}{p}]$ commutes with the operation of taking ext-groups in the category of isocrystals. This yields the desired result. □

3.4. **The local monodromy theorem.** We are ready to put all the previous results together and prove Theorem II. Let X be a smooth irreducible variety over a perfect field and let x be a closed point of X. In this section we view $X^{/x}$ as the scheme $\operatorname{Spec} \widehat{\mathcal{O}}_{X,x}$ (conventions of Notation 2.2.5 are in force). We denote by k the function field of X and by k_x the function field of $X^{/x}$. We also write η^{sep} (resp. $\bar{\eta}$) for the points over the generic point of X associated to a separable (resp. algebraic) closure of k.

Lemma 3.4.1. *The fields k and k_x have a common finite p-basis. In particular, $k \subseteq k_x$ is a separable field extension.*

Proof. By [58, Theorem 26.7], it is enough to show that $\Omega^1_{k/\mathbb{F}_p} \otimes_k k_x = \Omega^1_{k_x/\mathbb{F}_p}$. Write A for the local ring of X at x and $A_x^\wedge$ for the completion with respect to the maximal ideal $\mathfrak{m}_x$. Since A is regular, we can use [58, Theorem 30.5 and Theorem 30.9] to deduce that $\Omega^1_{A/\mathbb{F}_p} \otimes_A A_x^\wedge = \Omega^1_{A_x^\wedge/\mathbb{F}_p}$. We get the desired result after inverting $\mathfrak{m}_x - \{0\}$. □

Proposition 3.4.2. *If $(\mathcal{M}, \Phi_\mathcal{M})$ is an F-isocrystal over X such that $R_u(G(\mathcal{M}, \bar{\eta})) \subseteq U_\nu$, then $G(\mathcal{M}_{\eta^{\text{sep}}}, \bar{\eta}) = R_u(G(\mathcal{M}, \bar{\eta}))$.*

Proof. By Theorem 3.2.8 we have that $G(\mathcal{M}, \bar{\eta}) = G(\mathcal{M}_\eta, \bar{\eta})$, so that we are reduced to prove the statement for $G(\mathcal{M}_\eta, \bar{\eta})$. Note that the cover $f : \eta^{\text{sep}} \to \eta$ is a pro-étale $\text{Gal}(k^{\text{sep}}/k)$-cover and $\text{Gr}_{S_\bullet}(f^*\mathcal{M}_\eta)$ is trivial because η^{sep} is simply connected. This shows that we can apply Proposition 3.3.2 and deduce the desired result. □

Proposition 3.4.3. *If $(\mathcal{M}, \Phi_{\mathcal{M}})$ is an F-isocrystal over X coming from an irreducible overconvergent F-isocrystal with constant Newton polygon, then*

$$H^0_{\mathrm{cris}}(X^{/x}, (S_1(\mathcal{M}))^{/x}) = H^0_{\mathrm{cris}}(X^{/x}, \mathcal{M}^{/x}).$$

Proof. By Galois descent we may assume that the ring of constants of X is an algebraically closed field. The inclusion

$$H^0_{\mathrm{cris}}(X^{/x}, (S_1(\mathcal{M}))^{/x}) \subseteq H^0_{\mathrm{cris}}(X^{/x}, \mathcal{M}^{/x})$$

is an inclusion of F-isocrystals over κ. We suppose by contradiction that this is not an equality. Let $s_r > s_1$ be the greatest slope appearing in $H^0_{\mathrm{cris}}(X^{/x}, \mathcal{M}^{/x})$ and let v be a non-zero vector such that $\Phi^n_{\mathcal{M}^{/x}}(v) = p^{s_r n} v$ for $n \gg 0$. Write $(\tilde{\mathcal{M}}, \Phi_{\tilde{\mathcal{M}}})$ for the base change of $(\mathcal{M}, \Phi_{\mathcal{M}})$ to η^{sep}.

By the parabolicity conjecture, [22], $R_u(G(\mathcal{M}, \bar{\eta}))$ is contained in U_ν because $(\mathcal{M}, \Phi_{\mathcal{M}})$ comes from an irreducible overconvergent F-isocrystal. Proposition 3.4.2 then implies that the monodromy group $G(\tilde{\mathcal{M}}, \bar{\eta})$ is equal to $G(\mathcal{M}, \bar{\eta}) \cap U_\nu$. Therefore, the line spanned by v determines a rank 1 subobject $\tilde{\mathcal{L}} \subseteq S_r(\tilde{\mathcal{M}})/S_{r-1}(\tilde{\mathcal{M}})$ stabilised by the Frobenius. The preimage of this isocrystal in $S_r(\tilde{\mathcal{M}})$, denoted by $\tilde{\mathcal{N}}$, is kept invariant by the Frobenius and sits in an exact sequence

$$0 \to S_{r-1}(\tilde{\mathcal{M}}) \to \tilde{\mathcal{N}} \to \tilde{\mathcal{L}} \to 0.$$

Since $(\mathcal{M}, \Phi_{\mathcal{M}})$ comes from an irreducible overconvergent F-isocrystal, the sequence does not admit a Frobenius-equivariant splitting by [22, Theorem 4.1.3]. By Proposition 3.3.5, the base change of this extension to k_x^{sep} does not split as well. This leads to a contradiction since v is a vector in $H^0(X^{/x}, \mathcal{M}^{/x})$ which produces a non-trivial global section of $\tilde{\mathcal{N}} \subseteq \tilde{\mathcal{M}}$. □

We write η_x for the generic point of $X^{/x}$ and $G(\mathcal{M}^{/x}, \eta_x^{\mathrm{perf}})$ for the monodromy group of $\mathcal{M}^{/x}$ (notation as in §2.2.5) with respect to the perfection of η_x.

Theorem 3.4.4. *If $(\mathcal{M}, \Phi_{\mathcal{M}})$ comes from a semi-simple overconvergent F-isocrystal with constant Newton polygon, then*

$$G(\mathcal{M}^{/x}, \eta_x^{\mathrm{perf}}) = R_u(G(\mathcal{M}, \eta_x^{\mathrm{perf}})).$$

Proof. Write G for the group $G(\mathcal{M}, \eta)$ and V for the induced G-representation. By [22] we have that $R_u(G)$ is contained in U_ν where ν is the Newton cocharacter. Since $X^{/x}$ is geometrically simply connected we deduce that $\mathrm{Gr}_{S_\bullet}(\mathcal{M})^{/x}$ is trivial. This implies that $G(\mathcal{M}^{/x}, \eta_x^{\mathrm{perf}}) \subseteq R_u(G) \subseteq U_\nu$. Therefore, in order to apply the criterion of Proposition 3.1.2 it is enough to show that for every $\mathcal{N} \in \langle \mathcal{M} \rangle$, the space of global sections of $\mathcal{N}^{/x}$ is the same as the fibre at x of some direct sum of isoclinic subobjects of $\mathcal{N}$. To prove this, we may assume that $\mathcal{N}$ can be endowed with a Frobenius structure $\Phi_{\mathcal{N}}$ and $(\mathcal{N}, \Phi_{\mathcal{N}})$ is irreducible. Thanks to Proposition 3.4.3, we deduce that the fibre of $S_1(\mathcal{N})$ at x is the same as $H^0_{\mathrm{cris}}(X^{/x}, \mathcal{N}^{/x})$. This yields the desired result. □

4. Automorphism groups of p-divisible groups and Dieudonné–Lie algebras

The goal of this section is to define various groups of tensor preserving automorphisms and endomorphisms of p-divisible groups with G-structure, correcting some definitions from [47]. To do this, we introduce the notion of *Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra* s and we prove various properties using this point of view. We end the section by studying actions of algebraic groups on nilpotent Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra s and their associated unipotent groups.

4.1. **Hom groups of p-divisible groups.** We mainly follow [14, Section 3], [7, Section 4], and [47]. For p-divisible groups Y and Z over a perfect field κ, Chai and Oort construct finite group schemes

$$\mathbf{Hom}^{\mathrm{st}}(Y[p^n], Z[p^n]) \subseteq \mathbf{Hom}(Y[p^n], Z[p^n]),$$

where $\mathbf{Hom}(Y[p^n], Z[p^n])$ is the sheaf of homomorphisms from $Y[p^n]$ to $Z[p^n]$. They also construct maps

$$\pi_n : \mathbf{Hom}^{\mathrm{st}}(Y[p^n], Z[p^n]) \to \mathbf{Hom}^{\mathrm{st}}(Y[p^{n+1}], Z[p^{n+1}])$$

such that (the additive group underlying)

$$\varinjlim_n \mathbf{Hom}^{\mathrm{st}}(Y[p^n], Z[p^n])$$

is a p-divisible group $\mathcal{H}_{Y,Z}$, called the *internal-Hom p-divisible group.* Its scheme-theoretic p-adic Tate module is isomorphic to the group scheme $\mathbf{Hom}(Y, Z)$ of homomorphisms from Y to Z by [7, Lemma 4.1.7]. It follows from [*ibid.*, Lemma 4.1.8] that there is an isomorphism

$$\mathbb{D}(\mathcal{H}_{Y,Z})[\tfrac{1}{p}] = \underline{\mathrm{Hom}}(\mathbb{D}(Y)[\tfrac{1}{p}], \mathbb{D}(Z)[\tfrac{1}{p}])_{\leq 0},$$

where $(-)^{\leq 0}$ denotes the operation of taking the subspace of slope ≤ 0 of an F-isocrystal, and

$$\underline{\mathrm{Hom}}(\mathbb{D}(Y)[\tfrac{1}{p}], \mathbb{D}(Z)[\tfrac{1}{p}]).$$

denotes the internal-Hom in the category of F-isocrystals. By the proof of Lemma 4.1.8 of [*ibid.*] there is a canonical isomorphism of formal group schemes

$$\widetilde{\mathcal{H}}_{Y,Z} = \mathbf{Hom}(Y, Z)[\tfrac{1}{p}],$$

where $\widetilde{\mathcal{H}}_{Y,Z}$ is the universal cover of $\mathcal{H}_{Y,Z}$ in the sense of Scholze–Weinstein (see Section 2.3).

4.1.1. We will mostly be interested in $\mathcal{H}_Y := \mathcal{H}_{Y,Y}$ for a p-divisible group Y, in which case $T_p\mathcal{H}_Y = \mathbf{Hom}(Y, Y)$ and $\widetilde{\mathcal{H}}_Y = \mathbf{Hom}(\tilde{Y}, \tilde{Y})$ have an algebra structure. Up to isogeny, we can write $Y = \bigoplus_i Y_i$ as a direct sum of isoclinic p-divisible groups with slopes in increasing order. Then we can write endomorphisms of $\tilde{Y}$ in "block matrix form" to get the decomposition

$$\widetilde{\mathcal{H}}_Y = \bigoplus_{i,j} \widetilde{\mathcal{H}}_{Y_i,Y_j}.$$

It follows from [7, Lemma 4.1.8 and Corollary 4.1.10] that the p-divisible groups $\mathcal{H}_{Y_i,Y_j}$ are isoclinic and that: They are zero when $i > j$, étale p-divisible groups when $i = j$, and connected p-divisible groups when $i < j$. This means that we get a lower triangular block matrix form (see the proof of Proposition 4.2.11 of [*ibid.*]), and that the connected part

$$\widetilde{\mathcal{H}}_Y^{\circ} = \bigoplus_{i<j} \widetilde{\mathcal{H}}_{Y_i,Y_j}$$

consists of nilpotent endomorphisms. The étale part, instead, is precisely the locally profinite group scheme associated to the $\mathbb{Q}_p$-algebra given by $\mathbf{End}(Y)(\kappa)[\frac{1}{p}]$.

Lemma 4.1.2. *If $\kappa = \overline{\mathbb{F}}_p$ and $Y = \bigoplus_i Y_i$ is a direct sum of isoclinic p-divisible groups, then $\mathcal{H}_Y$ is completely slope divisible.*

Proof. The lemma follows from Lemma 2.3.14 since by [7, Lemma 4.1.8] each $\mathcal{H}_{Y_i,Y_j}$ is isoclinic. □

4.1.3. From now on we let $\kappa = \overline{\mathbb{F}}_p$. In order to work with Shimura varieties, it will be more fruitful to use the commutator bracket on $\mathbf{Hom}(Y,Y)[\frac{1}{p}]$ to equip $\widetilde{\mathcal{H}}_Y$ with the structure of a Lie algebra. More precisely, this is a Lie algebra over the locally profinite ring-scheme $\underline{\mathbb{Q}_p}$ associated to the topological ring $\mathbb{Q}_p$[3].

4.1.4. *Automorphisms.* Let $\mathbf{Aut}(\tilde{Y})$ be the functor on $\mathrm{Alg}^{\mathrm{op}}_{\overline{\mathbb{F}}_p}$ of automorphisms of $\tilde{Y}$. Then there is a monomorphism

$$\begin{aligned}\alpha : \mathbf{Aut}(\tilde{Y}) &\to \left(\widetilde{\mathcal{H}}_Y\right)^{\oplus 2}\\ \gamma &\mapsto (\gamma, \gamma^{-1})\end{aligned}$$

and the image consists of those pairs (γ, γ') such that $\gamma \circ \gamma' = 1 = \gamma' \circ \gamma$. It follows from this that α is representable in closed immersions and therefore $\mathbf{Aut}(\tilde{Y})$ is a formal scheme. If we intersect with the subscheme $\mathbf{Hom}(Y,Y)^{\oplus 2} = T_p\mathcal{H}_Y^{\oplus 2}$ we recover the group scheme $\mathbf{Aut}(Y)$. The projection to the diagonal map

$$\mathbf{Aut}(\tilde{Y}) \to \underline{\mathbf{Aut}(\tilde{Y})(\overline{\mathbb{F}}_p)},$$

has connected kernel $\mathbf{Aut}(\tilde{Y})^\circ$. This implies that we have a semi-direct product decomposition

$$\mathbf{Aut}(\tilde{Y}) \simeq \mathbf{Aut}(\tilde{Y})^\circ \rtimes \underline{\mathbf{Aut}(\tilde{Y})(\overline{\mathbb{F}}_p)}.$$

4.1.5. Our block matrix description of $\mathrm{End}(\tilde{Y})$ implies that all elements γ of $\mathbf{Aut}(\tilde{Y})^\circ$ are unipotent automorphisms of $\tilde{Y}$. This means that the logarithm map

$$\begin{aligned}L : \mathbf{Aut}(\tilde{Y})^\circ &\to \widetilde{\mathcal{H}}_Y^\circ\\ X &\mapsto \sum_{i=1}^{\infty}(-1)^{i+1}\tfrac{(X-1)^i}{i}\end{aligned}$$

and the exponential map

$$\begin{aligned}E : \widetilde{\mathcal{H}}_Y^\circ &\to \mathbf{Aut}(\tilde{Y})^\circ\\ X &\mapsto \sum_{i=0}^{\infty}\tfrac{X^i}{i!}\end{aligned}$$

are well defined and we have $E \circ L = 1$ and $L \circ E = 1$.

Remark 4.1.6. In particular, $\mathbf{Aut}(\tilde{Y})^\circ$ is isomorphic as a functor to $\widetilde{\mathcal{H}}_Y^\circ$, thus it is representable by a formal scheme that is a filtered colimit of spectra of qrsp rings.

The Baker–Campbell–Hausdorff (BCH) formula gives an expression for the group structure on $\mathbf{Aut}(\tilde{Y})^\circ$ in terms of the Lie bracket on $\widetilde{\mathcal{H}}_Y^\circ$ (see [71, Part I, Chapter IV, §7-8]). In fact if V is a (possibly infinite dimensional) vector space over a field of characteristic zero and X, Y are two nilpotent endomorphisms of V, then the BCH formula expresses $\exp(X)\exp(Y)$ in terms of X and Y and their iterated Lie brackets.

[3]If V is a topological space we use the notation $\underline{V}$ for the functor on $\overline{\mathbb{F}}_p$-schemes sending $T \mapsto \mathrm{Hom}_{\mathrm{cont}}(|T|, V)$, where $|T|$ is the topological space underlying the scheme T. The functor $\underline{V}$ is representable by a finite scheme if V is finite and discrete, and therefore also representable if T is profinite or locally profinite.

It follows from this that the filtration $\mathrm{Fil}^\bullet$ induces a filtration $\mathrm{Fil}^\bullet_{\mathbf{Aut}} = E(\mathrm{Fil}^\bullet)$ of $\mathbf{Aut}(\tilde{Y})^\circ$ by normal subgroups, with graded quotients isomorphic to

$$\bigoplus_i \widetilde{\mathcal{H}}_{Y_i, Y_{i+k}}.$$

Indeed, we can identify the graded pieces of the two filtrations via the exponential map, and the BCH formula tells us that the exponential map of an abelian Lie algebra is an isomorphism of groups.

4.2. **Dieudonné–Lie algebras.** This section is an interlude on Dieudonné modules endowed with a Lie bracket. Recall that $\breve{\mathbb{Z}}_p = W(\overline{\mathbb{F}}_p)$, let $\breve{\mathbb{Q}}_p = \breve{\mathbb{Z}}_p[\frac{1}{p}]$ and write φ for the Frobenius on both $\breve{\mathbb{Z}}_p$ and $\breve{\mathbb{Q}}_p$.

Definition 4.2.1. A *Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra* is a triple $(\mathfrak{a}^+, \varphi_{\mathfrak{a}^+}, [-,-])$ where $(\mathfrak{a}^+, \varphi_{\mathfrak{a}^+})$ is a Dieudonné module over $\overline{\mathbb{F}}_p$ (see Definition 2.3.1) and

$$[-,-] : \mathfrak{a}^+ \times \mathfrak{a}^+ \to \mathfrak{a}^+$$

is a Lie bracket such that the following diagram commutes

$$\begin{array}{ccc} \mathfrak{a}^+[\frac{1}{p}] \times \mathfrak{a}^+[\frac{1}{p}] & \xrightarrow{\varphi_{\mathfrak{a}^+} \times \varphi_{\mathfrak{a}^+}} & \mathfrak{a}^+[\frac{1}{p}] \times \mathfrak{a}^+[\frac{1}{p}] \\ \downarrow{\scriptstyle [-,-]} & & \downarrow{\scriptstyle [-,-]} \\ \mathfrak{a}^+[\frac{1}{p}] & \xrightarrow{\varphi_{\mathfrak{a}^+}} & \mathfrak{a}^+[\frac{1}{p}]. \end{array}$$

A morphism of Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebras is a $\breve{\mathbb{Z}}_p$-linear map $f : \mathfrak{a}^+ \to \mathfrak{b}^+$ that respects the Lie brackets and induces a homomorphism of Dieudonné modules. If f is injective with finite cokernel we say that f is an *isogeny*. We write $\mathbb{X}(\mathfrak{a}^+)$ for the p-divisible group attached to the Dieudonné module $(\mathfrak{a}^+, \varphi_{\mathfrak{a}^+})$. We also say that a Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra is *completely slope divisible* if the underlying Dieudonné module is so (see Definition 2.3.12).

Similarly, a *Dieudonné–Lie $\breve{\mathbb{Q}}_p$-algebra* is a triple $(\mathfrak{a}, \varphi_{\mathfrak{a}}, [-,-])$ where $(\mathfrak{a}, \varphi_{\mathfrak{a}})$ is a rational Dieudonné module over $\overline{\mathbb{F}}_p$ and $[-,-]$ is a Lie bracket of $\mathfrak{a}$ satisfying the same compatibility. We write

$$(\mathfrak{a}, \varphi_{\mathfrak{a}}, [-,-])$$

for the Dieudonné–Lie $\breve{\mathbb{Q}}_p$-algebra obtained from a Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra

$$(\mathfrak{a}^+, \varphi_{\mathfrak{a}^+}, [-,-])$$

by inverting p. We also denote by $\tilde{\mathbb{X}}(\mathfrak{a})$ the universal cover of the p-divisible group associated to an integral lattice of $(\mathfrak{a}, \varphi_{\mathfrak{a}})$. This assignment does not depend on the choice of the lattice. We will very often omit the Frobenius structure and the Lie bracket in the notation of Dieudonné–Lie algebras.

Remark 4.2.2. Alternatively, Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebras (resp. $\breve{\mathbb{Q}}_p$-algebras) can be defined as the Lie algebra objects in the symmetric monoidal category of F-crystals (resp. F-isocrystals) over $\overline{\mathbb{F}}_p$ such that the underlying F-crystal (resp. F-isocrystal) is a Dieudonné module.

Example 4.2.3. The first example of a Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra is the Dieudonné module of the internal-Hom p-divisible group $\mathcal{H}_Y$ attached to a p-divisible group Y over $\overline{\mathbb{F}}_p$, denoted by $\mathbb{D}(\mathcal{H}_Y)$. Indeed, the Lie bracket coming from the commutator bracket on $T_p\mathcal{H}_Y = \mathbf{Hom}(Y, Y)$, induces an φ-equivariant Lie bracket on $\mathfrak{a}$ by [40, Corollary 1.2.5]. The Lie bracket on $\mathbf{Hom}(Y, Y)$ clearly sends the identity component $\mathbf{Hom}(Y,Y)^\circ$ to itself. This leads to our second example of a Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra, the one induced on $\mathbb{D}(\mathcal{H}_Y^\circ)$.

Given a Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra, there is also a natural procedure to get many other isogenous Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebras by using the Frobenius structure. Let us see this more in detail, as it will play an important role.

Construction 4.2.4. For a Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra $(\mathfrak{a}^+, \varphi_{\mathfrak{a}^+}, [-,-])$ and $n \in \mathbb{Z}$, we define $\Phi^n(\mathfrak{a}^+)$ to be the $\breve{\mathbb{Z}}_p$-submodule $\varphi_{\mathfrak{a}^+}^n(\mathfrak{a}^+) \subseteq \mathfrak{a}$. This $\breve{\mathbb{Z}}_p$-submodule is preserved by the Frobenius and the Lie bracket of $\mathfrak{a}$. We then get on $\Phi^n(\mathfrak{a}^+)$ a Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra structure. Note that $\varphi_{\mathfrak{a}}^{-n}$ induces an isomorphism $\Phi^n(\mathfrak{a}^+) \xrightarrow{\sim} \mathfrak{a}^{+,(n)}$ of Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebras.

Lemma 4.2.5. *For $n \geq 0$, there is a natural exact sequence of fpqc sheaves*

$$0 \to T_p\mathbb{X}(\mathfrak{a}^+) \to T_p\mathbb{X}(\Phi^n(\mathfrak{a}^+)) \to \mathbb{X}(\mathfrak{a}^+)[F^n] \to 0.$$

Proof. Thanks to the isomorphism $\varphi_{\mathfrak{a}}^{-n} \colon \Phi^n(\mathfrak{a}^+) \xrightarrow{\sim} \mathfrak{a}^{+,(n)}$, this is equivalent to proving that we have an exact sequence

$$0 \to T_p\mathbb{X}(\mathfrak{a}^+) \to T_p\mathbb{X}(\mathfrak{a}^+)^{(n)} \to \mathbb{X}(\mathfrak{a}^+)[F^n] \to 0,$$

where $T_p\mathbb{X}(\mathfrak{a}^+) \to T_p\mathbb{X}(\mathfrak{a}^+)^{(n)}$ is induced by the nth-power of the (relative) Frobenius of $\mathbb{X}(\mathfrak{a}^+)$. The result then follows by a classical diagram chase, using the fact that the Frobenius of $\mathbb{X}(\mathfrak{a}^+)$ is faithfully flat. □

The fundamental lemma we will use very often to study Dieudonné–Lie $\breve{\mathbb{Q}}_p$-algebras and reduce ourself to the abelian case is the following one.

Lemma 4.2.6. *Let $\mathfrak{a}$ be a Dieudonné–Lie $\breve{\mathbb{Q}}_p$-algebra where all the slopes are negative. Let μ_1 be the smallest slope of $\mathfrak{a}$ and let $\mathfrak{b} \subseteq \mathfrak{a}$ be an F-stable $\breve{\mathbb{Q}}_p$-subspace that is isoclinic of that slope. Then $\mathfrak{b}$ is contained in the centre of $\mathfrak{a}$.*

Proof. There are no nonzero F-equivariant maps $\mathfrak{b} \otimes \mathfrak{a} \to \mathfrak{a}$ because, by assumption, all the slopes of $\mathfrak{b} \otimes \mathfrak{a}$ are strictly smaller than the slopes of $\mathfrak{a}$. Hence the restriction of the Lie bracket to $\mathfrak{b} \times \mathfrak{a}$ is trivial. □

4.2.7. *Integrability.* In general, for Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebras, the BCH formula is not well-defined. We want to clarify here how to bypass this issue in our context. Let us first recall the formula as presented in [71, Part I, Chapter IV, §7-8]. We focus on the *nilpotent* setting, since it is the only one we will need.

Definition 4.2.8. We say that a Dieudonné–Lie $\breve{\mathbb{Q}}_p$-algebra $(\mathfrak{a}, \varphi_{\mathfrak{a}}, [-,-])$ is *nilpotent* if the underlying Lie algebra $(\mathfrak{a}, [-,-])$ is nilpotent. We also say that a Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra is *nilpotent* if the associated Dieudonné–Lie $\breve{\mathbb{Q}}_p$-algebra is so.

For positive integers d, n, we denote by $\Delta_n(d) \subseteq \mathbb{N}^n \times \mathbb{N}^n$ the subset of elements $(\underline{r}, \underline{s}) \in \mathbb{N}^n \times \mathbb{N}^n$ with $\underline{r} = (r_1, \dots, r_n)$ and $\underline{s} = (s_1, \dots, s_n)$ such that $\sum_{i=1}^n (r_i + s_i) = d$ and $r_i + s_i \neq 0$ for every i. The BCH series can be written as

$$\mathrm{BCH}(X,Y) \coloneqq \sum_{d=1}^{\infty} \sum_{n=1}^{\infty} \sum_{(\underline{r},\underline{s}) \in \Delta_n(d)} (-1)^{n-1} \frac{[X^{r_1} Y^{s_1} \cdots X^{r_n} Y^{s_n}]}{dn \prod_{i=1}^n r_i! s_i!}.$$

If $\mathfrak{a}$ is a nilpotent Dieudonné–Lie $\breve{\mathbb{Q}}_p$-algebra, then for every $a, b \in \mathfrak{a}$, there is a well-defined element $\mathrm{BCH}(a,b) \in \mathfrak{a}$. Indeed, for d and n big enough the terms of the series vanish. For (Dieudonné–)Lie $\breve{\mathbb{Z}}_p$-algebras, instead, the series might not be defined when the denominators are too divisible by p. It then makes sense to consider the following definition.

Definition 4.2.9. We say that a nilpotent Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra is *integrable* if for every $a, b \in \mathfrak{a}^+$, each summand

$$(-1)^{n-1}\frac{[a^{r_1}b^{s_1}\cdots a^{r_n}b^{s_n}]}{dn\prod_{i=1}^n r_i!s_i!}$$

of $\mathrm{BCH}(a,b)$ lies in $\mathfrak{a}^+$.

Lemma 4.2.10. *If $\mathfrak{a}^+$ is a nilpotent Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra, then $p^2\mathfrak{a}^+$ is an integrable Dieudonné–Lie $\breve{\mathbb{Z}}_p$-subalgebra.*

Proof. Write $\mathfrak{b}^+$ for the Dieudonné–Lie $\breve{\mathbb{Z}}_p$-subalgebra $p^2\mathfrak{a}^+ \subseteq \mathfrak{a}^+$. We have to prove that for every $v, w \in \mathfrak{b}^+$, each term

$$(-1)^{n-1}\frac{[v^{r_1}w^{s_1}\cdots v^{r_n}w^{s_n}]}{dn\prod_{i=1}^n r_i!s_i!}$$

lies in $\mathfrak{b}^+$. By the bilinearity of the Lie bracket, we have that $[\mathfrak{b}^+, \mathfrak{b}^+] \subseteq p^2\mathfrak{b}^+$. Thus by induction, we deduce that for every $e \geq 2$ and every set of elements $\{v_1, \ldots, v_e\}$ in $\mathfrak{b}^+$, the nested bracket $[v_1 \cdots v_e]$ lies in $p^{2e-2}\mathfrak{b}^+$. The nested bracket $[v^{r_1}w^{s_1}\cdots v^{r_n}w^{s_n}]$ is then an element of $p^{2d-2}\mathfrak{b}^+$. Since the denominator $dn\prod_{i=1}^n r_i!s_i!$ divides $(d!)^2$, we get the desired result thanks to the fact that $\frac{p^{d-1}}{d!}$ is an element of $\breve{\mathbb{Z}}_p$. □

4.2.11. *Dieudonné–theory and bilinear maps.* Let $(\mathfrak{a}^+, \varphi_{\mathfrak{a}^+}, [-,-])$ be a Dieudonné–Lie algebra and let $\mathbb{X}(\mathfrak{a}^+)$ be the unique p-divisible group over $\overline{\mathbb{F}}_p$ with (covariant) Dieudonné module $(\mathfrak{a}^+, \varphi_{\mathfrak{a}^+})$. We want to equip its universal cover and its Tate-module with a bilinear map, coming from the Lie bracket on $\mathfrak{a}^+$. For this we record a result that tells us how Dieudonné-theory for universal covers of p-divisible groups interacts with $\mathbb{Q}_p$-bilinear maps. The analogous result for bihomomorphisms of finite flat group schemes is [40, Corollary 1.2.5].

Lemma 4.2.12. *Let Y_1, Y_2, Y_3 be p-divisible groups over $\overline{\mathbb{F}}_p$, then there is a functorial and $\mathbb{Q}_p$-linear bijection between the space of $\breve{\mathbb{Q}}_p$-bilinear maps*

$$g : \mathbb{D}(Y_1)[\tfrac{1}{p}] \times \mathbb{D}(Y_2)[\tfrac{1}{p}] \to \mathbb{D}(Y_3)[\tfrac{1}{p}]$$

that satisfy $g(\varphi_{Y_1}x, \varphi_{Y_2}y) = \varphi_{Y_3}(g(x,y)$ and the space of bilinear maps

$$f : \tilde{Y}_1 \times \tilde{Y}_2 \to \tilde{Y}_3.$$

Moreover if $Y_1 = Y_2 = Y_3$, then f satisfies the Jacobi identity if and only if g does.

Proof. Recall that the internal-Hom p-divisible group $\mathcal{H}_{Y_2,Y_3}$ satisfies

$$\widetilde{\mathcal{H}}_{Y_2,Y_3} = \mathbf{Hom}(Y_2, Y_3)[\tfrac{1}{p}] = \mathbf{Hom}(\tilde{Y}_2, \tilde{Y}_3).$$

It follows from the usual tensor-hom adjunction for $\mathbb{Q}_p$-vector spaces that bilinear maps

$$\tilde{Y}_1 \times \tilde{Y}_2 \to \tilde{Y}_3$$

are in bijection with homomorphisms

$$\tilde{Y}_1 \to \widetilde{\mathcal{H}}_{Y_2,Y_3}.$$

It also follows from the tensor-hom adjunction in the category of F-isocrystals that $\breve{\mathbb{Q}}_p$-bilinear maps

$$\mathbb{D}(Y_1)[\tfrac{1}{p}] \times \mathbb{D}(Y_2)[\tfrac{1}{p}] \to \mathbb{D}(Y_3)[\tfrac{1}{p}]$$

that commute with the Frobenius as above are in bijection with morphisms of F-isocrystals

$$\mathbb{D}(Y_1)[\tfrac{1}{p}] \to \underline{\mathrm{Hom}}(\mathbb{D}(Y_2)[\tfrac{1}{p}], \mathbb{D}(Y_3)[\tfrac{1}{p}]),$$

where $\underline{\mathrm{Hom}}$ denotes the internal-Hom in the category of F-isocrystals. Since the slope of $\mathbb{D}(Y_1)[\frac{1}{p}]$ is bounded above by 0, these are also in bijection with morphisms of F-isocrystals

$$\mathbb{D}(Y_1)[\tfrac{1}{p}] \to \underline{\mathrm{Hom}}(\mathbb{D}(Y_2)[\tfrac{1}{p}], \mathbb{D}(Y_3)[\tfrac{1}{p}])_{\leq 0}.$$

By [7, Lemma 4.1.7] there is an isomorphism

$$\mathbb{D}(\mathcal{H}_{Y_2,Y_3})[\tfrac{1}{p}] = \underline{\mathrm{Hom}}(\mathbb{D}(Y_2)[\tfrac{1}{p}], \mathbb{D}(Y_3)[\tfrac{1}{p}])_{\leq 0}$$

and Dieudonné-theory over $\overline{\mathbb{F}}_p$ tells us that homomorphisms

$$\tilde{Y}_1 \to \widetilde{\mathcal{H}}_{Y_2,Y_3}$$

are in bijection with morphisms of F-isocrystals

$$\mathbb{D}(Y_1)[\tfrac{1}{p}] \to \mathbb{D}(\mathcal{H}_{Y_2,Y_3})[\tfrac{1}{p}],$$

which is what we wanted to prove. Similarly, we can prove a correspondence for trilinear maps. Since the Jacobi identity can be expressed as the vanishing of a trilinear map, this concludes the proof of the lemma. □

Remark 4.2.13. The correspondence between bilinear maps of rational Dieudonné modules and bilinear maps of universal covers of p-divisible groups can be described explicitly on R-points for qrsp R using the isomorphism

$$\tilde{Y}_i(R) = \left(B^+_{\mathrm{cris}}(R) \otimes_{\breve{\mathbb{Q}}_p} \mathbb{D}(Y_i)[\tfrac{1}{p}]\right)^{\varphi=1}$$

of Lemma 2.3.10. Indeed, a $\breve{\mathbb{Q}}_p$-bilinear map

$$\mathbb{D}(Y_1)[\tfrac{1}{p}] \times \mathbb{D}(Y_2)[\tfrac{1}{p}] \to \mathbb{D}(Y_3)[\tfrac{1}{p}]$$

that commutes with the Frobenius as above induces a $B^+_{\mathrm{cris}}(R)$-bilinear map

$$B^+_{\mathrm{cris}}(R) \otimes_{\breve{\mathbb{Q}}_p} \mathbb{D}(Y_1)[\tfrac{1}{p}] \times B^+_{\mathrm{cris}}(R) \otimes_{\breve{\mathbb{Q}}_p} \mathbb{D}(Y_2)[\tfrac{1}{p}] \to B^+_{\mathrm{cris}}(R) \otimes_{\breve{\mathbb{Q}}_p} \mathbb{D}(Y_3)[\tfrac{1}{p}]$$

that induces a $\mathbb{Q}_p$-bilinear map on the $\varphi = 1$ subspaces.

Corollary 4.2.14. *Suppose that we are given a $\breve{\mathbb{Z}}_p$-bilinear map*

$$g^+ : \mathbb{D}(Y_1) \times \mathbb{D}(Y_2) \to \mathbb{D}(Y_3)$$

satisfying $g^+(\varphi_{Y_1}(x), \varphi_{Y_2}(y)) = \varphi_{Y_3}(g^+(x,y))$. Then the induced map $f : \tilde{Y}_1 \times \tilde{Y}_2 \to \tilde{Y}_3$ restricts to a $\mathbb{Z}_p$-bilinear Lie bracket

$$f^+ : T_pY_1 \times T_pY_2 \to T_pY_3.$$

Proof. It suffices to show this on R-valued points for semiperfect R and in fact, by Yoneda, it suffices to check it in the universal case when R is the ring underlying $T_pY_1 \times T_pY_2$. But this R is qrsp, so we can use Lemma 2.3.10 which tells us that

$$T_pY_i(R) = \left(\mathrm{A}_{\mathrm{cris}}(R) \otimes_{\breve{\mathbb{Z}}_p} \mathbb{D}(Y_i)\right)^{\varphi=1},$$

and we see that f^+ sends $T_pY_1(R) \times T_pY_2(R)$ to $T_pY_3(R)$ because g^+ sends $\mathbb{D}(Y_1) \times \mathbb{D}(Y_2)$ to $\mathbb{D}(Y_3)$. □

4.3. **Formal homogeneous spaces.** In this section we focus on the fundamental constructions obtained from the datum of a nilpotent Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra. They will be widely used in the text to describe the infinitesimal behaviour of central leaves of Shimura varieties.

Construction 4.3.1. Let $\mathfrak{a}$ be a nilpotent Dieudonné–Lie $\breve{\mathbb{Q}}_p$-algebra. By Lemma 4.2.12, the Lie bracket on $\mathfrak{a}$ induces a Lie bracket on $\tilde{\mathbb{X}}(\mathfrak{a})$. The BCH formula defines then a formal group structure

$$m_{\text{Lie}} \colon \tilde{\mathbb{X}}(\mathfrak{a}) \times \tilde{\mathbb{X}}(\mathfrak{a}) \to \tilde{\mathbb{X}}(\mathfrak{a}).$$

We define $\tilde{\Pi}(\mathfrak{a})$ to be the formal group $(\tilde{\mathbb{X}}(\mathfrak{a}), m_{\text{Lie}})$. The assignment $\mathfrak{a} \mapsto \tilde{\Pi}(\mathfrak{a})$ defines a functor

$$\tilde{\Pi} \colon \left\{\text{Nilpotent Dieudonné–Lie } \breve{\mathbb{Q}}_p\text{-algebras}\right\} \to \left\{\text{Formal groups over } \overline{\mathbb{F}}_p\right\}.$$

Construction 4.3.2. If $\mathfrak{a}^+$ is an integrable nilpotent Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra, by Corollary 4.2.14, the Lie bracket on $\mathfrak{a}^+$ induces a Lie bracket on $T_p\mathbb{X}(\mathfrak{a}^+)$. The formal group structure m_{Lie} on $\tilde{\mathbb{X}}(\mathfrak{a})$ restricts then to a group scheme structure on $T_p\mathbb{X}(\mathfrak{a}^+)$. We denote by $\Pi(\mathfrak{a}^+)$ the group scheme $(T_p\mathbb{X}(\mathfrak{a}^+), m_{\text{Lie}})$. Note that m_{Lie} preserves $p^mT_p\mathbb{X}(\mathfrak{a}^+)$ for every $m \geq 0$, so that

$$\Pi(\mathfrak{a}^+) = \varprojlim_{m \geq 0} \Pi_m(\mathfrak{a}^+),$$

where $\Pi_m(\mathfrak{a}^+)$ is the affine finite scheme $\mathbb{X}(\mathfrak{a}^+)[p^m]$ endowed with the group scheme structure induced by m_{Lie}. This shows that $\Pi(\mathfrak{a}^+)$ is a *profinite group scheme*[4].

In this case, the assignment $\mathfrak{a}^+ \mapsto \Pi(\mathfrak{a}^+)$ defines a functor

$$\Pi \colon \left\{\text{Integrable nilpotent Dieudonné–Lie } \breve{\mathbb{Z}}_p\text{-algebras}\right\} \to \left\{\text{Profinite group schemes over } \overline{\mathbb{F}}_p\right\}.$$

Remark 4.3.3. If the slopes of $\mathfrak{a}$ are negative, then $\tilde{\Pi}(\mathfrak{a})$ is a connected affine formal group. Similarly, in the integral situation, if the slopes of $\mathfrak{a}^+$ are negative, then $\Pi(\mathfrak{a}^+)$ is a connected profinite group scheme.

To continue our analysis, it will be convenient to work under additional assumptions on $\mathfrak{a}^+$.

Definition 4.3.4. A *plain Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra* is a completely slope divisible integrable Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra $\mathfrak{a}^+$ such that all the slopes are negative. For every slope μ, we write $\mathfrak{a}^+_\mu \subseteq \mathfrak{a}^+$ for the $\breve{\mathbb{Z}}_p$-subspace of slope μ. Note that if μ_1 is the minimal slope of $\mathfrak{a}^+$, then $\mathfrak{a}^+_{\mu_1} \subseteq \mathfrak{a}^+$ is a central plain Dieudonné–Lie $\breve{\mathbb{Z}}_p$-subalgebra by Lemma 4.2.6 and Lemma 2.3.15.

As we have seen in Construction 4.2.4, for every $\mathfrak{a}^+$ and $n \in \mathbb{Z}$, there is a Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra $\Phi^n(\mathfrak{a}^+)$, constructed using the Frobenius, which is isogenous to $\mathfrak{a}^+$. It is easy to check that if $\mathfrak{a}^+$ is plain, then $\Phi^n(\mathfrak{a}^+)$ is plain for every n.

Construction 4.3.5. Let $\mathfrak{a}^+$ be a plain Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra. For every $n \geq 0$, we write $\Pi^n(\mathfrak{a}^+)$ for $\Pi(\Phi^n(\mathfrak{a}^+))$ and, for $m \geq 0$, we write $\Pi^n_m(\mathfrak{a}^+)$ for $\Pi_m(\Phi^n(\mathfrak{a}^+))$. There are natural maps $\alpha_n \colon \Pi_n(\mathfrak{a}^+) \to \Pi^n_n(\mathfrak{a}^+)$. We define $Z^n(\mathfrak{a}^+)$ to be the fppf-quotient

$$\Pi^n_n(\mathfrak{a}^+)/\alpha_n(\Pi_n(\mathfrak{a}^+))$$

[4] We recall that a *profinite group scheme* over $\overline{\mathbb{F}}_p$ is a group scheme over $\overline{\mathbb{F}}_p$ that is the inverse limit of finite (not necessarily commutative) group schemes over $\overline{\mathbb{F}}_p$.

over $\mathrm{Alg}_{\overline{\mathbb{F}}_p}^{\mathrm{op}}$. We also write $Z(\mathfrak{a}^+)$ for the fppf-sheaf obtained as the inductive limit

$$\varinjlim_n Z^n(\mathfrak{a}^+).$$

There is a natural action of $\tilde{\Pi}(\mathfrak{a})$ on the formal scheme $Z(\mathfrak{a}^+)$ and an equivariant map $\tilde{\Pi}(\mathfrak{a}) \to Z(\mathfrak{a}^+)$. We say that $Z(\mathfrak{a}^+)$ is the *formal homogeneous space associated to* $\mathfrak{a}^+$.

Remark 4.3.6. By Lemma 4.2.5, if $\mathfrak{a}^+$ is abelian, then $Z^n(\mathfrak{a}^+) = \mathbb{X}(\mathfrak{a}^+)[F^n]$ and $Z(\mathfrak{a}^+) = \mathbb{X}(\mathfrak{a}^+)$. In general, the formal scheme $Z(\mathfrak{a}^+)$ is the fpqc quotient $\tilde{\Pi}(\mathfrak{a})/\Pi(\mathfrak{a}^+)$ (see Lemma 4.3.12).

We want to prove a fundamental representability result for $Z(\mathfrak{a}^+)$. For this we use a construction that allows us to reduce many statements to the case when $\mathfrak{a}^+$ is abelian.

Construction 4.3.7. Let $\mathfrak{a}^+$ be a plain Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra and let μ_1 be the minimal slope of $\mathfrak{a}^+$. The formal group structure m_{Lie} induces a morphism of affine schemes $\Pi_n^n(\mathfrak{a}_{\mu_1}^+) \times \Pi_n^n(\mathfrak{a}^+) \to \Pi_n^n(\mathfrak{a}^+)$. By Lemma 4.2.6, the group $\tilde{\Pi}(\mathfrak{a}_{\mu_1})$ is in the centre of $\tilde{\Pi}(\mathfrak{a})$, thus we also get a morphism

$$\mathbb{X}(\mathfrak{a}_{\mu_1}^+)[F^n] \times Z^n(\mathfrak{a}^+) \to Z^n(\mathfrak{a}^+)$$

which endows $Z^n(\mathfrak{a}^+)$ with a left action of $\mathbb{X}(\mathfrak{a}_{\mu_1}^+)[F^n]$. This makes $Z^n(\mathfrak{a}^+)$ an fppf-torsor over $Z^n(\mathfrak{a}^+/\mathfrak{a}_{\mu_1}^+)$ under the finite syntomic group scheme $\mathbb{X}(\mathfrak{a}_{\mu_1}^+)[F^n]$.

Proposition 4.3.8. *If $\mathfrak{a}^+$ is a plain Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra, then for every $n \geq 1$ the fppf-sheaf $Z^n(\mathfrak{a}^+)$ is represented by* $\operatorname{Spec} R_{n,m}$ *where m is the dimension of $\mathbb{X}(\mathfrak{a}^+)$ and*

$$R_{n,m} := \overline{\mathbb{F}}_p[X_1, \cdots, X_m]/(X_1^{p^n}, \cdots, X_m^{p^n}).$$

Moreover, the torsor $Z^n(\mathfrak{a}^+) \to Z^n(\mathfrak{a}^+/\mathfrak{a}_{\mu_1}^+)$ of Construction 4.3.7 is trivial.

Proof. We want to prove the result by induction on the number of slopes of $\mathfrak{a}^+$. In the isoclinic case the result follows from [60, Proposition 2.1.2]. For the inductive step, we first note that by Construction 4.3.7, the fppf-sheaf $Z^n(\mathfrak{a}^+)$ is represented by a connected scheme which is finite and syntomic over $Z^n(\mathfrak{a}^+/\mathfrak{a}_{\mu_1}^+)$. The result is then obtained as a consequence of Proposition 3.6.8 of [27, Chapter III]. □

Corollary 4.3.9. *If $\mathfrak{a}^+$ is a plain Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra, then $Z(\mathfrak{a}^+)$ is a formal Lie variety of the same dimension as $\mathbb{X}(\mathfrak{a}^+)$.*

By Corollary 4.3.9 we get a functor

$$Z \colon \left\{\text{Plain Dieudonné–Lie } \breve{\mathbb{Z}}_p\text{-algebras}\right\} \to \left\{\text{Formal Lie varieties over } \overline{\mathbb{F}}_p\right\}.$$

Lemma 4.3.10. *The natural map $\tilde{\Pi}(\mathfrak{a}) \to Z(\mathfrak{a}^+)$ is an fpqc-torsor under the affine group scheme $\Pi(\mathfrak{a}^+)$.*

Proof. Thanks to Lemma 4.2.5, for every $n \geq 0$ we have a cartesian diagram

$$\begin{array}{ccc} \Pi(\mathfrak{a}^+) & \longrightarrow & \Pi^n(\mathfrak{a}^+) \\ \downarrow & \square & \downarrow \\ \alpha_n(\Pi_n(\mathfrak{a}^+)) & \longrightarrow & \Pi_n^n(\mathfrak{a}^+). \end{array}$$

By diagram chasing, we deduce that $\Pi^n(\mathfrak{a}^+) \to Z^n(\mathfrak{a}^+)$ is a torsor for $\Pi(\mathfrak{a}^+)$. This yields the desired result. □

Lemma 4.3.11. *If X is an fpqc sheaf, $P \to X$ is an fpqc-torsor under the affine group scheme $\Pi(\mathfrak{a}^+)$ and $P \to \tilde{\Pi}(\mathfrak{a})$ is a $\Pi(\mathfrak{a}^+)$-equivariant map, then there is a unique induced map $X \to Z(\mathfrak{a}^+)$ lying in the commutative diagram*

$$\begin{array}{ccc} P & \longrightarrow & \tilde{\Pi}(\mathfrak{a}) \\ \downarrow & & \downarrow \\ X & \dashrightarrow & Z(\mathfrak{a}^+). \end{array}$$

Proof. If $x : \operatorname{Spec} R \to X$ is an R-point of X, then we need to produce an R-point $y : \operatorname{Spec} R \to Z(\mathfrak{a}^+)$. After passing to the fpqc cover $\operatorname{Spec} R \times_X P =: \operatorname{Spec} R' \to \operatorname{Spec} R$, we can lift x to a $\operatorname{Spec} R'$-point $\tilde{x}$ of P. We then let y' be the image of $\tilde{x}$ in $Z(\mathfrak{a}^+)(R')$. We need to show that y' descends to an R-point y of $Z(\mathfrak{a}^+)$. Since the latter is an fqpc-sheaf, by Corollary 4.3.9 we simply need to show that the two pullbacks of y' to $\operatorname{Spec} R'' := \operatorname{Spec} R' \times_{\operatorname{Spec} R} \operatorname{Spec} R'$ agree. But $\operatorname{Spec} R'' = \Pi(\mathfrak{a}^+) \times \operatorname{Spec} R$, and the condition on the two pullbacks comes down to asking that y' is fixed by the action of $\Pi(\mathfrak{a}^+)$. This follows from Lemma 4.3.10. □

Lemma 4.3.12. *Let α be an ordinal as in* [76, Lemma 000J] *and let $\mathrm{Alg}^{\mathrm{op}}_{\overline{\mathbb{F}}_p,\alpha} \subseteq \mathrm{Alg}^{\mathrm{op}}_{\overline{\mathbb{F}}_p}$ be the subcategory of algebras of cardinality at most α. If*

$$\tilde{\Pi}(\mathfrak{a})/_{\alpha}\Pi(\mathfrak{a}^+)$$

is the quotient as an fpqc sheaf over $\mathrm{Alg}^{\mathrm{op}}_{\overline{\mathbb{F}}_p,\alpha}$, then $\tilde{\Pi}(\mathfrak{a})/_{\alpha}\Pi(\mathfrak{a}^+) = Z(\mathfrak{a}^+)$. In particular, the quotient is independent of α.

Proof. Arguing as in Lemma 4.3.11, since $\tilde{\Pi}(\mathfrak{a}) \to Z(\mathfrak{a}^+)$ is a $\Pi(\mathfrak{a}^+)$-torsor of fpqc sheaves over $\mathrm{Alg}^{\mathrm{op}}_{\overline{\mathbb{F}}_p,\alpha}$, there is a canonical map $\tilde{\Pi}(\mathfrak{a})/_{\alpha}\Pi(\mathfrak{a}^+) \to Z(\mathfrak{a}^+)$ factorising the morphism $\tilde{\Pi}(\mathfrak{a}) \to Z(\mathfrak{a}^+)$. Both $\tilde{\Pi}(\mathfrak{a}) \to \tilde{\Pi}(\mathfrak{a})/_{\alpha}\Pi(\mathfrak{a}^+)$ and $(\tilde{\Pi}(\mathfrak{a})/_{\alpha}\Pi(\mathfrak{a}^+)) \times_{Z(\mathfrak{a}^+)} \tilde{\Pi}(\mathfrak{a}) \to \tilde{\Pi}(\mathfrak{a})/_{\alpha}\Pi(\mathfrak{a}^+)$ are fpqc torsors under $\Pi(\mathfrak{a}^+)$, thus the diagram

$$\begin{array}{ccc} \tilde{\Pi}(\mathfrak{a}) & \xrightarrow{=} & \tilde{\Pi}(\mathfrak{a}) \\ \downarrow & & \downarrow \\ \tilde{\Pi}(\mathfrak{a})/_{\alpha}\Pi(\mathfrak{a}^+) & \longrightarrow & Z(\mathfrak{a}^+) \end{array}$$

is cartesian. Thanks to the fact that $\tilde{\Pi}(\mathfrak{a}) \to Z(\mathfrak{a}^+)$ is a surjection of fpqc sheaves, we deduce that $\tilde{\Pi}(\mathfrak{a})/_{\alpha}\Pi(\mathfrak{a}^+) \to Z(\mathfrak{a}^+)$ is an isomorphism, as we wanted. □

Lemma 4.3.13. *For every plain Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra $\mathfrak{a}^+$, the natural map $\tilde{\Pi}(\mathfrak{a}) \to Z(\mathfrak{a}^+)$ induces an isomorphism*

$$\tilde{\Pi}(\mathfrak{a}) \simeq Z(\mathfrak{a}^+)^{\mathrm{perf}}.$$

Proof. For every $n \geq 0$, we have the following factorisation of the nth-power of the absolute Frobenius of $Z(\mathfrak{a}^+)$

$$\begin{array}{ccc} Z(\mathfrak{a}^+) & \xrightarrow{F^n} & Z(\mathfrak{a}^+) \\ & \searrow^{\sim} \quad \nearrow & \\ & Z(\Phi^{-n}(\mathfrak{a}^+)), & \end{array}$$

where $Z(\Phi^{-n}(\mathfrak{a}^+)) \to Z(\mathfrak{a}^+)$ is induced by the natural inclusion $\Phi^{-n}(\mathfrak{a}^+) \subseteq \mathfrak{a}^+$. This implies that

$$Z(\mathfrak{a}^+)^{\mathrm{perf}} = \varprojlim_n Z(\Phi^{-n}(\mathfrak{a}^+)).$$

Since $\bigcap_n \Phi^{-n}(\mathfrak{a}^+) = 0$, we deduce that $\tilde{\Pi}(\mathfrak{a}) \to \varprojlim_n Z(\Phi^{-n}(\mathfrak{a}^+))$ is a monomorphism, thus by Lemma 2.1.2 it is a closed immersion. On the other hand, $\tilde{\Pi}(\mathfrak{a}) \to \varprojlim_n Z(\Phi^{-n}(\mathfrak{a}^+))$ is flat since for every $n \geq 0$ the map $\tilde{\Pi}(\mathfrak{a}) \to Z(\Phi^{-n}(\mathfrak{a}^+))$ is so. This implies the desired result. □

4.4. **Automorphism groups for Shimura varieties of Hodge type.** Let Y be a p-divisible group over $\overline{\mathbb{F}}_p$ and fix an isomorphism $\mathbb{D}(Y)[\frac{1}{p}] \simeq V \otimes_{\mathbb{Q}_p} \breve{\mathbb{Q}}_p$, where V is a vector space over $\mathbb{Q}_p$. In order to generalise Section 4.1 to Shimura varieties of Hodge type, we let $b \in \mathrm{GL}_V(\breve{\mathbb{Q}}_p)$ be the Frobenius of the F-isocrystal $\mathbb{D}(Y)[\frac{1}{p}]$. Then the internal-Hom F-isocrystal

$$\underline{\mathrm{Hom}}(\mathbb{D}(Y)[\tfrac{1}{p}], \mathbb{D}(Y)[\tfrac{1}{p}])$$

is isomorphic to the F-isocrystal (where $\mathrm{Ad}\,\sigma b$ is the Frobenius given by the adjoint action of $b \in \mathrm{GL}_V(\breve{\mathbb{Q}}_p)$ on $\mathfrak{gl}_V$)

$$(\mathfrak{gl}_V \otimes \breve{\mathbb{Q}}_p, \mathrm{Ad}\,\sigma\, b).$$

If we are given a reductive group $G \subseteq \mathrm{GL}_V$ such that $b \in G(\breve{\mathbb{Q}}_p)$, then we get an inclusion of F-isocrystals

$$(\mathfrak{g} \otimes \breve{\mathbb{Q}}_p, \mathrm{Ad}\,\sigma\, b) \subseteq (\mathfrak{gl}_V \otimes \breve{\mathbb{Q}}_p, \mathrm{Ad}\,\sigma\, b).$$

The slope filtration of this F-isocrystal is described in [47, Section 3]: The slope t part is given by

$$\bigoplus_{\substack{\alpha_0 \in \Phi_0^+ \\ \langle \alpha_0, \nu_b \rangle = t}} \mathfrak{u}_{\alpha_0},$$

where α_0 runs over the relative roots of $G_{\breve{\mathbb{Q}}_p}$ with respect to a maximal $\breve{\mathbb{Q}}_p$-split torus S defined over $\mathbb{Q}_p$ that is contained in a Borel subgroup B with respect to which the Newton cocharacter ν_b of b is dominant (see [54, Section 1.1.2] for the definition of the Newton cocharacter). We see that the slope ≤ 0 part corresponds precisely to the Lie algebra of the standard parabolic subgroup $P_b = P_{\nu_b}$ associated to ν_b and that the slope 0 part corresponds to the Lie algebra of the Levi M_b.

Taking non-positive slope parts we get an F-stable sub-isocrystal

$$\mathrm{Lie}\, P_{\nu_b} \subseteq \mathbb{D}(\mathcal{H}_Y)[\tfrac{1}{p}]$$

and intersecting with $\mathbb{D}(\mathcal{H}_Y)$ we get a Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra which we write as $\mathbb{D}(\mathcal{H}_Y^G)$ for a p-divisible group $\mathcal{H}_Y^G \subseteq \mathcal{H}_Y$. Note that by construction and Corollary 4.2.14,

$$T_p\mathcal{H}_Y^G \subseteq T_p\mathcal{H}_Y = \mathbf{Hom}(Y, Y)$$

is stable under the commutator bracket.

Corollary 4.4.1. *The dimension of $\mathcal{H}_Y^G$ is equal to $\langle 2\rho, \nu_b \rangle$.*

Proof. This is explained at the end of the proof of [47, Proposition 3.1.4]. □

Thanks to Lemma 4.2.12, we have that

$$\widetilde{\mathcal{H}}_Y^G \subseteq \widetilde{\mathcal{H}}_Y = \mathbf{Hom}(Y,Y)[\tfrac{1}{p}]$$

is closed under the Lie bracket (the commutator) of $\mathbf{Hom}(Y,Y)[\frac{1}{p}]$. The Dieudonné module

$$\mathbb{D}(\mathcal{H}_Y^{G,\circ})[\tfrac{1}{p}]$$

is stable under the Lie bracket and consists precisely of the strictly negative slope part of the isocrystal $\operatorname{Lie} P_{\nu_b}$, which can be identified with

$$\operatorname{Lie} U_{\nu_b} \subseteq \operatorname{Lie} P_{\nu_b},$$

where U_{ν_b} is the unipotent radical of P_{ν_b}.

Corollary 4.4.2. *If Y is completely slope divisible, then $\mathcal{H}_Y^G$ is completely slope divisible.*

Proof. This follows from Lemma 2.3.15 and Lemma 4.1.2. □

4.4.3. Let us fix some notation to define the group of automorphisms of $\tilde{Y}$ which preserve the G-structure. Here we use the convention that for an object M in a rigid category we write $M^{\otimes}$ for the direct sum of $M^{\otimes n} \otimes (M^*)^{\otimes m}$ for all pairs of integers $m \geq 0, n \geq 0$.

As in [54, Section 1.3.4], we can choose a collection of tensors $\{s_\alpha\}_{\alpha \in \mathscr{A}} \subseteq V^{\otimes}$ such that G is their pointwise stabiliser in GL_V. If we identify the Lie algebra $\mathfrak{gl}_V$ of GL_V with the vector space of endomorphisms of V, then by [42, Lemma 5.3.3] $\mathfrak{g} \subseteq \mathfrak{gl}_V$ consists of those endomorphisms g satisfying $g^{\otimes} s_\alpha = 0$, let us call such endomorphisms *tensor annihilating endomorphisms*. It follows that

$$\mathbb{D}(\mathcal{H}_Y^{G,\circ})[\tfrac{1}{p}] \subseteq \mathbb{D}(\mathcal{H}_Y)[\tfrac{1}{p}]$$

is the subspace of $\mathbb{D}(\mathcal{H}_Y)[\frac{1}{p}]$ of tensor annihilating endomorphisms. Therefore by Remark 4.2.13 it follows that for qrsp $\overline{\mathbb{F}}_p$-algebras R we have that

$$\widetilde{\mathcal{H}}_Y^G(R) \subseteq \widetilde{\mathcal{H}}_Y(R) = \operatorname{Hom}(Y_R, Y_R)[\tfrac{1}{p}]$$

consists of the endomorphisms $\tilde{Y}_R \to \tilde{Y}_R$ such that the induced endomorphism

$$\mathbb{D}(g) : \mathbb{D}(Y)[\tfrac{1}{p}] \otimes_{\breve{\mathbb{Q}}_p} B^+_{\mathrm{cris}}(R) \to \mathbb{D}(Y)[\tfrac{1}{p}] \otimes_{\breve{\mathbb{Q}}_p} B^+_{\mathrm{cris}}(R)$$

satisfies $g^{\otimes}(s_\alpha \otimes 1) = 0$ for all α. It follows that

$$T_p\mathcal{H}_Y^G(R) \subseteq T_p\mathcal{H}_Y(R) = \operatorname{Hom}(Y_R, Y_R)$$

consists precisely of the endomorphisms $g : Y_R \to Y_R$ such that the induced endomorphism

$$\mathbb{D}(g) : \mathbb{D}(Y) \otimes_{\breve{\mathbb{Z}}_p} A_{\mathrm{cris}}(R) \to \mathbb{D}(Y) \otimes_{\breve{\mathbb{Z}}_p} A_{\mathrm{cris}}(R)$$

satisfies $g^{\otimes}(s_\alpha \otimes 1) = 0$ for all α. In both cases we will use the term *tensor annihilating endomorphisms* to denote such endomorphisms of $\tilde{Y}_R$ or Y_R. We use also the notion of *tensor preserving automorphism* of $\tilde{Y}_R$ which is an automorphism g of $\tilde{Y}_R$ such that the induced automorphism

$$\mathbb{D}(g) : \mathbb{D}(Y)[\tfrac{1}{p}] \otimes_{\breve{\mathbb{Q}}_p} B^+_{\mathrm{cris}}(R) \to \mathbb{D}(Y)[\tfrac{1}{p}] \otimes_{\breve{\mathbb{Q}}_p} B^+_{\mathrm{cris}}(R)$$

satisfies $g^{\otimes}(s_\alpha \otimes 1) = (s_\alpha \otimes 1)$ for all α.

Lemma 4.4.4. *There is a (unique) closed subgroup*

$$\mathbf{Aut}_G(\tilde{Y}) \subseteq \mathbf{Aut}(\tilde{Y})$$

such that on qrsp $\overline{\mathbb{F}}_p$-algebras R the subgroup

$$\mathbf{Aut}_G(\tilde{Y})(R) \subseteq \mathbf{Aut}(\tilde{Y})(R)$$

consists precisely of the tensor preserving automorphisms. Moreover, there is an isomorphism of formal groups

$$\mathbf{Aut}_G(\tilde{Y}) \simeq \mathbf{Aut}_G(\tilde{Y})^\circ \rtimes \underline{J_b(\mathbb{Q}_p)},$$

where $\mathbf{Aut}_G(\tilde{Y})^\circ$ is the intersection of $\mathbf{Aut}(\tilde{Y})^\circ$ with $\mathbf{Aut}_G(\tilde{Y})$ inside $\mathbf{Aut}(\tilde{Y})$ and $J_b(\mathbb{Q}_p) \subseteq G(\breve{\mathbb{Q}}_p)$ is the twisted centraliser of b.

Proof. It is clear from the definition that the exponential of a nilpotent tensor annihilating endomorphism of $\tilde{Y}_R$ is a unipotent tensor preserving automorphism of $\tilde{Y}_R$. Conversely the logarithm of a unipotent tensor preserving automorphism of $\tilde{Y}_R$ is a nilpotent tensor annihilating endomorphism of $\tilde{Y}_R$. Since $\mathbf{Aut}(\tilde{Y})^\circ \subseteq \mathbf{Aut}(\tilde{Y})$ is precisely the subgroup of unipotent automorphisms, the exponential map defines an isomorphism of functors

$$\mathcal{H}_Y^{G,\circ} \simeq \mathbf{Aut}_G(\tilde{Y})^\circ,$$

and thus $\mathbf{Aut}_G(\tilde{Y})^\circ \subseteq \mathbf{Aut}(\tilde{Y})^\circ$ is representable in closed immersions. We have seen that there is a semi-direct product decomposition

$$\mathbf{Aut}(\tilde{Y}) \simeq \mathbf{Aut}(\tilde{Y})^\circ \rtimes \underline{\mathbf{Aut}(\tilde{Y})(\overline{\mathbb{F}}_p)}.$$

There is a closed subgroup of the locally profinite group $\underline{\mathbf{Aut}(\tilde{Y})(\overline{\mathbb{F}}_p)}$ consisting of those automorphisms of $\tilde{Y}$ that are tensor preserving. By Dieudonné theory we can identify this group with the group of tensor preserving automorphisms of the F-isocrystal $\mathbb{D}(Y)[\frac{1}{p}]$; this group is precisely the twisted centraliser $J_b(\mathbb{Q}_p) \subseteq G(\breve{\mathbb{Q}}_p)$ of b.

The action of $J_b(\mathbb{Q}_p) \subseteq G(\breve{\mathbb{Q}}_p)$ on $\operatorname{Lie} G$ stabilises $\operatorname{Lie} U_{\nu_b}$ since $J_b(\mathbb{Q}_p)$ is contained in the centraliser of ν_b inside $G(\breve{\mathbb{Q}}_p)$. Therefore we get a closed subgroup

$$\mathbf{Aut}_G(\tilde{Y})^\circ \rtimes \underline{J_b(\mathbb{Q}_p)} \subseteq \mathbf{Aut}(\tilde{Y})^\circ \rtimes \underline{\mathbf{Aut}_G(\tilde{Y})(\overline{\mathbb{F}}_p)},$$

whose R-points for qrsp R give the group of tensor preserving automorphisms of $\tilde{Y}_R$. ☐

Our construction does *not* agree with [47, Definition 2.3.1], which defines $\mathbf{Aut}_G(\tilde{Y})$ as the intersection of

$$\left(\widetilde{\mathcal{H}}_Y^G \times \widetilde{\mathcal{H}}_Y^G\right) \cap \mathbf{Aut}(\tilde{Y}). \tag{4.4.1}$$

By the discussion above, the R-points of this functor are given by automorphisms g of $\tilde{Y}_R$ such that the induced automorphism

$$\mathbb{D}(g) : \mathbb{D}(Y)[\tfrac{1}{p}] \otimes_{\breve{\mathbb{Q}}_p} B^+_{\mathrm{cris}}(R) \to \mathbb{D}(Y)[\tfrac{1}{p}] \otimes_{\breve{\mathbb{Q}}_p} B^+_{\mathrm{cris}}(R)$$

satisfies $g^{\otimes}(s_\alpha \otimes 1) = 0$ for all α. Note that an automorphism g of $\tilde{Y}_R$ induces an automorphism of $\mathbb{D}(Y)[\frac{1}{p}] \otimes_{\breve{\mathbb{Q}}_p} B^+_{\mathrm{cris}}(R)$. Thus if g is an R-point of this intersection then $g^{\otimes}(s_\alpha \otimes 1)$ cannot be zero unless $s_\alpha \otimes 1 = 0$ which implies that $s_\alpha = 0$. Therefore the functor (4.4.1) is empty unless $G = \mathrm{GL}_V$.

Fortunately, the rest of [47] only uses the characterisation of $\mathbf{Aut}_G(\tilde{Y})$ of Lemma 4.4.4. In particular, [47, Proposition 3.2.4] is correct as stated. Therefore the rest of [47] is not affected.

We end this section by defining

$$\mathbf{Aut}_G(Y) = \mathbf{Aut}_G(\tilde{Y}) \times_{\mathbf{Aut}(\tilde{Y})} \mathbf{Aut}(Y)$$

so that

$$\mathbf{Aut}_G(Y)^\circ = \mathbf{Aut}_G(\tilde{Y})^\circ \times_{\mathbf{Aut}(\tilde{Y})^\circ} \mathbf{Aut}(Y)^\circ$$

Remark 4.4.5. If $\mathbb{D}(\mathcal{H}_Y^\circ)$ is plain (for example if $p \gg 0$), then we can identify the group schemes

$$\Pi(\mathbb{D}(\mathcal{H}_Y^\circ)) \simeq \mathbf{Aut}_G(Y)^\circ,$$

but in general there is no containment in either direction.

4.5. **Strongly nontrivial actions.** Let $(\mathfrak{a}^+, \varphi_{\mathfrak{a}^+}, [-,-])$ be a plain Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra with associated Dieudonné–Lie $\breve{\mathbb{Q}}_p$-algebra $(\mathfrak{a}, \varphi_{\mathfrak{a}}, [-,-])$. Let $\mathbf{Aut}(\mathfrak{a}, \varphi_{\mathfrak{a}})$ be the automorphism group of the underlying F-isocrystal, considered as an algebraic group over $\mathbb{Q}_p$ (as in [69, Section 1.1]). Then the Lie algebra of $\mathbf{Aut}(\mathfrak{a}, \varphi_{\mathfrak{a}})$ can be identified with the endomorphism algebra of the F-isocrystal, equipped with the commutator bracket.

There is a closed subgroup

$$\mathbf{Aut}(\mathfrak{a}, \varphi_{\mathfrak{a}}, [-,-]) \subseteq \mathbf{Aut}(\mathfrak{a}, \varphi_{\mathfrak{a}})$$

consisting of those automorphisms preserving the Lie bracket.

Definition 4.5.1. Let Q be an algebraic group over $\mathbb{Q}_p$ equipped with a group homomorphism

$$Q \to \mathbf{Aut}(\mathfrak{a}, \varphi_{\mathfrak{a}}, [-,-]).$$

We call such a group homomorphism (or action) *strongly non-trivial* if the induced linear representation of Q on $\mathfrak{a}$ does not have trivial subquotients.

Example 4.5.2. If $\mathfrak{a} = \operatorname{Lie} U_{\nu_b}$ for some element $b \in G(\breve{\mathbb{Q}}_p)$ admissible with respect to some Shimura datum on G, then the algebraic group J_b has a natural and strongly non-trivial action on $\mathfrak{a}$. Moreover, by Proposition 5.5.1 of [42] the restriction of this action to a maximal torus $T \subseteq J_b$ is still strongly non-trivial.

5. Formal neighbourhoods of central leaves

5.1. **Introduction.** In this section we will recall the constructions of the canonical integral models of Shimura varieties of Hodge type at hyperspecial level from [51]. We then recall the definitions of central leaves $C_{\mathsf{G},[\![b]\!]}$ inside the special fibers $\mathrm{Sh}_{\mathsf{G},U}$ of these canonical integral models, and also the definition of Igusa varieties from [37, 38, 47]. The main goal of this section is to study the structure of the formal completions $C_{\mathsf{G},[\![b]\!]}^{/x}$ of central leaves at points $x \in C_{\mathsf{G},[\![b]\!]}(\overline{\mathbb{F}}_p)$. We will show, using the work of Caraiani–Scholze and Kim, that the perfection of the formal scheme $C_{\mathsf{G},[\![b]\!]}^{/x}$ has a canonical (generally non-commutative) group structure. These groups are of the form $\tilde{\Pi}(\mathfrak{a})$ for a Dieudonné–Lie $\breve{\mathbb{Q}}_p$-algebra $\mathfrak{a}$, as introduced in Section 4.

In Section 5.5 we will show that F-stable Lie subalgebras $\mathfrak{b} \subseteq \mathfrak{a}$ give rise to formally smooth closed subschemes of $C_{\mathsf{G},[\![b]\!]}^{/x}$. These are precisely the *strongly Tate-linear* subspaces of Chai and Oort. We end by stating a conjecture on the monodromy group of the universal isocrystal over such subspaces.

5.2. **Integral models.** For a symplectic space (V,ψ) over $\mathbb{Q}$, we write $\mathsf{G}_V := \mathrm{GSp}(V,\psi)$ for the group of symplectic similitudes of V over $\mathbb{Q}$. It admits a Shimura datum $(\mathsf{G}_V,\mathsf{H}_V)$, where H_V is the union of the Siegel upper and lower half-spaces. Let (G,X) be a Shimura datum of Hodge type with reflex field E and let $(\mathsf{G},\mathsf{X}) \to (\mathsf{G}_V,\mathsf{H}_V)$ be a Hodge embedding.

Fix a prime $p>2$ and a place v above p of the reflex field E, and write $G = \mathsf{G}\otimes\mathbb{Q}_p$, write $E = \mathsf{E}_v$ for the v-adic completion of E and $\mathcal{O}_E$ for its ring of integers. Let $K_p \subseteq G(\mathbb{Q}_p)$ be a hyperspecial subgroup. Then after possibly changing the Hodge embedding and the symplectic space V, we can find a self dual $\mathbb{Z}_{(p)}$-lattice $V_{(p)}$, such that K_p is the stabiliser in $G(\mathbb{Q}_p)$ of $V_p := V_{(p)} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p$, see Section 2.3.15 of [50]. Write $G_{\mathbb{Z}_{(p)}}$ for the Zariski closure of G in $GL(V_{\mathbb{Z}_{(p)}})$, then $G_{\mathbb{Z}_{(p)}} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p$ is a reductive integral model $\mathcal{G}$ of G.

For every sufficiently small compact open subgroup $U^p \subseteq \mathsf{G}(\mathbb{A}_f^p)$, we can find $\mathcal{U}^p \subseteq \mathsf{G}_V(\mathbb{A}_f^p)$ such that the Hodge embedding induces a closed immersion (see Lemma 2.1.2 of [51])

$$\mathbf{Sh}_U(\mathsf{G},\mathsf{X}) \to \mathbf{Sh}_{\mathcal{U}}(\mathsf{G}_V,\mathsf{H}_V) \otimes_{\mathbb{Q}} \mathsf{E}$$

of (canonical models of) Shimura varieties of levels $U = U^pU_p$ and $\mathcal{U} = \mathcal{U}^p\mathcal{U}_p$, respectively. We let $\mathcal{S}_{\mathcal{U}}(\mathsf{G}_V,\mathsf{H}_V)$ over $\mathbb{Z}_p$ be the moduli-theoretic integral model of $\mathbf{Sh}_{\mathcal{U}}(\mathsf{G}_V,\mathsf{H}_V)$; it is a moduli space of polarised abelian schemes (A,λ) up to prime-to-p isogeny with level $\mathcal{U}^p$-structure. Let

$$\mathscr{S}_U := \mathscr{S}_U(\mathsf{G},\mathsf{X}) \to \mathcal{S}_{\mathcal{U}}(\mathsf{G}_V,\mathsf{H}_V) \otimes_{\mathbb{Z}_p} \mathcal{O}_E$$

be the normalisation of the Zariski closure of $\mathbf{Sh}_U(\mathsf{G},\mathsf{X})$ in $\mathcal{S}_{\mathcal{U}}(\mathsf{G}_V,\mathsf{H}_V) \otimes_{\mathbb{Z}_p} \mathcal{O}_E$. Then by the main result of [51] and [49], see Section 2 of [49], the scheme $\mathbf{Sh}_U(\mathsf{G},\mathsf{X})$ is smooth and in fact isomorphic to the canonical integral model of $\mathbf{Sh}_U(\mathsf{G},\mathsf{X})$. The main result of [81] tells us that

$$\mathscr{S}_U \to \mathcal{S}_{\mathcal{U}} \otimes_{\mathbb{Z}_p} \mathcal{O}_{E,v}$$

is a closed immersion. Choose an algebraic closure $\overline{\mathbb{F}}_p$ of the residue field of $\mathcal{O}_E$ and let $\mathrm{Sh}_{\mathsf{G},U}$ be the base change to $\overline{\mathbb{F}}_p$ of $\mathscr{S}_U$. We will write $\mathrm{Sh}_{\mathsf{G}_V,\mathcal{U}}$ for the base change to $\overline{\mathbb{F}}_p$ of $\mathcal{S}_{\mathcal{U}}$. Then the pullback of the universal abelian scheme (up to prime-to-p isogeny) over $\mathcal{S}_{\mathcal{U}}$ gives rise to an abelian scheme (up to prime-to-p isogeny) A over $\mathrm{Sh}_{G,U}$ with associated p-divisible group $X = A[p^\infty]$.

5.2.1. *Tensors.* Recall the notation $V^{\otimes}_{\mathbb{Z}_{(p)}}$ from Section 4.4.3. By [51, Lemma 1.3.2], the subgroup $G_{\mathbb{Z}_{(p)}} \subseteq GL(V_{\mathbb{Z}_{(p)}})$ is the stabiliser of a collection of tensors $s_\alpha \in V^{\otimes}_{\mathbb{Z}_{(p)}}$.

For $x \in \mathrm{Sh}_{\mathsf{G},U}(\overline{\mathbb{F}}_p)$, we write A_x for the abelian variety up to prime-to-p isogeny over $\overline{\mathbb{F}}_p$ corresponding to the image of $x \in \mathrm{Sh}_{\mathsf{G}_V,\mathcal{U}}$. Let $x \in \mathrm{Sh}_{\mathsf{G},U}(\overline{\mathbb{F}}_p)$ and let $\mathbb{D}_{\mathrm{contr},x}$ be the (contravariant!)[5] Dieudonné module of $A_x[p^\infty]$. It is explained in [73, Section 6.3] that there are canonical tensors $\{s_{\alpha,\mathrm{cris},x}\}$ in

$$\mathbb{D}^{\otimes}_{\mathrm{contr},x},$$

that are invariant under the Frobenius on $\mathbb{D}_{\mathrm{contr},x}[\frac{1}{p}]$. It is moreover explained there that there is an isomorphism

$$\mathbb{D}_{\mathrm{contr},x} \simeq V_{(p)} \otimes_{\mathbb{Z}_{(p)}} \breve{\mathbb{Z}}_p \tag{5.2.1}$$

taking $s_{\alpha,\mathrm{cris},x}$ to $s_\alpha \otimes 1$. Under such an isomorphism, the Frobenius corresponds to an element $b_x \in G(\breve{\mathbb{Q}}_p)$, which is well defined up to σ-conjugacy by $\mathcal{G}(\breve{\mathbb{Z}}_p)$, where $\sigma : G(\breve{\mathbb{Q}}_p) \to G(\breve{\mathbb{Q}}_p)$ is

[5] As in [42], we use both covariant and contravariant Dieudonné theory in this paper. Since we mostly use the covariant theory, we will denote all our contravariant Dieudonné modules with the subscript $_{\mathrm{contr}}$. The reason for this is that all the work on integral models of Shimura varieties uses the contravariant theory, while results about internal-Hom p-divisible groups are best expressed in the covariant theory.

induced by the Frobenius $\sigma : \breve{\mathbb{Q}}_p \to \breve{\mathbb{Q}}_p$ (which we called φ before). We will denote by $[\![b_x]\!]$ the $\mathcal{G}(\breve{\mathbb{Z}}_p)$-$\sigma$-conjugacy class of b_x and by $[b_x]$ the $G(\breve{\mathbb{Q}}_p)$-σ-conjugacy class of b_x.

5.3. **Central leaves.** It follows from [38, Corollary 3.3.8] that for $b \in G(\breve{\mathbb{Q}}_p)$ there are (reduced) locally closed subschemes

$$C_{\mathsf{G},[\![b]\!]} \subseteq \mathrm{Sh}_{\mathsf{G},U,[b]} \subseteq \mathrm{Sh}_{\mathsf{G},U}$$

of $\mathrm{Sh}_{\mathsf{G},U}$ such that their $\overline{\mathbb{F}}_p$-points are given by

$$C_{\mathsf{G},[\![b]\!]}(\overline{\mathbb{F}}_p) = \{x \in \mathrm{Sh}_{\mathsf{G},U}(\overline{\mathbb{F}}_p) \ : \ [\![b_x]\!] = [\![b]\!]\}$$
$$\mathrm{Sh}_{\mathsf{G},U,[b]}(\overline{\mathbb{F}}_p) = \{x \in \mathrm{Sh}_{\mathsf{G},U}(\overline{\mathbb{F}}_p) \ : \ [b_x] = [b]\}.$$

The subscheme $\mathrm{Sh}_{\mathsf{G},U,[b]}$ is called the *Newton stratum* attached to $[b]$, and the subscheme $C_{\mathsf{G},[\![b]\!]} \subseteq \mathrm{Sh}_{\mathsf{G},U,[b]}$ is called the *central leaf* attached to $[\![b]\!]$. We note that the natural map $C_{\mathsf{G},[\![b]\!]} \to \mathrm{Sh}_{\mathsf{G},U,[b]}$ is a closed immersion by [38, Corollary 3.3.8] and that the central leaf $C_{\mathsf{G},[\![b]\!]}$ is smooth and equidimensional by [48, Corollary 5.3.1]. The following remark is [43, Remark 2.1.4].

Remark 5.3.1. When $(\mathsf{G}, \mathsf{X}) = (\mathsf{G}_V, \mathsf{H}_V)$, then the $\mathcal{G}(\breve{\mathbb{Z}}_p)$-conjugacy class $[\![b_x]\!]$ captures precisely the isomorphism class of the polarised p-divisible group $(A_x[p^\infty], \lambda_x)$, where an isomorphism of polarised p-divisible groups $f : (Y, \mu) \to (Y', \mu')$ is an isomorphism $f : Y \to Y'$ such that $f^*\mu' = c\mu$ for some $c \in \mathbb{Z}_p^\times$. In particular, when $(\mathsf{G}, \mathsf{X}) = (\mathsf{G}_V, \mathsf{H}_V)$ our central leaves do *not* agree with those defined in [16], which are defined using isomorphisms $f : (Y, \mu) \to (Y', \mu')$ with $f^*\mu' = \mu$.

Fix a point $x \in \mathrm{Sh}_{\mathsf{G},U}(\overline{\mathbb{F}}_p)$ and write $Y = A_x[p^\infty]$ and write λ for the induced polarisation. We write $[\![b]\!] := [\![b_x]\!]$ for the $\mathcal{G}(\breve{\mathbb{Z}}_p)$-$\sigma$-conjugacy class of elements of $G(\breve{\mathbb{Q}}_p)$ associated to x. We will write $C_{(Y,\lambda)} \subseteq \mathrm{Sh}_{\mathsf{G}_V,\mathcal{U}}$ for the central leaf associated to the polarised p-divisible group (Y, λ).

Hypothesis 5.3.2. The p-divisible group Y is completely slope divisible.

It follows from [47, Proposition 2.4.5] that for every Newton stratum $\mathrm{Sh}_{\mathsf{G},U,[b]} \subseteq \mathrm{Sh}_{\mathsf{G},U}$ we can always find a central leaf $C_{\mathsf{G},[\![b']\!]} \subseteq \mathrm{Sh}_{\mathsf{G},U,[b]}$ such that $C_{\mathsf{G},[\![b']\!]} \subseteq C_{(Y',\lambda')}$ where $C_{(Y',\lambda')}$ corresponds to a completely slope divisible p-divisible group Y'. Thus this is not an unreasonable assumption.

As explained in [56, Section 3.2.3], this implies that $X = A[p^\infty]$ over $C_{(Y,\lambda)}$ admits a slope filtration and we will denote the associated graded pieces for the slope filtration by X_i. We can then consider the Igusa towers

$$\mathrm{Ig}_{\mathrm{CS},\lambda} \to \mathrm{Ig}_{\mathrm{M},\lambda} \to C_{(Y,\lambda)},$$

where $\mathrm{Ig}_{\mathrm{M},\lambda} \to C_{(Y,\lambda)}$ is the moduli space, constructed by Mantovan, of isomorphisms

$$X_i \simeq Y_{i,C_{(Y,\lambda)}}$$

compatible with the polarisations up to a scalar in $\mathbb{Z}_p^\times$, and $\mathrm{Ig}_{\mathrm{CS},\lambda} \to C_{(Y,\lambda)}$ is the moduli space, constructed by Caraiani–Scholze, of isomorphisms

$$X \simeq Y_{C_{(Y,\lambda)}},$$

compatible with the polarisations up to a scalar in $\mathbb{Z}_p^\times$. Let $\mathbf{Aut}_\lambda(Y)$ be the group scheme of automorphisms of Y that preserve the polarisation λ up to a $\mathbb{Z}_p^\times$-scalar and let $\underline{\mathbf{Aut}_\lambda(Y)(\overline{\mathbb{F}}_p)}$ be its pro-étale group scheme of connected components.

It follows from work of Mantovan, [56], that $C_{(Y,\lambda)}$ is smooth and that $\mathrm{Ig}_{\mathrm{M},\lambda} \to C_{(Y,\lambda)}$ is a pro-étale torsor for $\underline{\mathbf{Aut}_\lambda(Y)(\overline{\mathbb{F}}_p)}$. Caraiani and Scholze, [7], prove that $\mathrm{Ig}_{\mathrm{CS},\lambda}$ is perfect and that the map $\mathrm{Ig}_{\mathrm{CS},\lambda} \to \mathrm{Ig}_{\mathrm{M},\lambda}$ identifies it with the perfection of $\mathrm{Ig}_{\mathrm{M},\lambda}$. This implies that $\mathrm{Ig}_{\mathrm{CS},\lambda} \to C_{(Y,\lambda)}$

is faithfully flat and thus a torsor for $\mathbf{Aut}_\lambda(Y)$. Moreover they prove that the action of $\mathbf{Aut}_\lambda(Y)$ on $\mathrm{Ig}_{\mathrm{CS},\lambda}$ extends to an action of $\mathbf{Aut}_\lambda(\tilde{Y})$, the group of automorphisms of $\tilde{Y}$ preserving the polarisation up to a $\mathbb{Q}_p^\times$-scalar.

We note that the following diagram is cartesian

$$\begin{array}{ccc} \mathrm{Ig}_{\mathrm{CS},\lambda} & \longrightarrow & \mathrm{Ig}_{\mathrm{M},\lambda} \\ \big\downarrow & & \big\downarrow \\ C^{\mathrm{perf}}_{(Y,\lambda)} & \longrightarrow & C_{(Y,\lambda)}, \end{array} \tag{5.3.1}$$

where $C^{\mathrm{perf}}_{(Y,\lambda)}$ denotes the perfection of $C_{(Y,\lambda)}$.

5.4. **Igusa varieties for Shimura varieties of Hodge type.** Let $x \in \mathrm{Sh}_{\mathsf{G},U}(\overline{\mathbb{F}}_p)$ as above and choose an isomorphism

$$\mathbb{D}_{\mathrm{contr},x} \simeq V_{(p)} \otimes_{\mathbb{Z}_{(p)}} \breve{\mathbb{Z}}_p$$

taking $s_{\alpha,\mathrm{cris},x}$ to $s_\alpha \otimes 1$ as in (5.2.1). This induces an isomorphism from $V^*_{(p)} \otimes \breve{\mathbb{Z}}_p$ to the covariant Dieudonné module $\mathbb{D}(Y)$ and thus gives us Frobenius invariant tensors $\{s_{\alpha,\mathrm{cris},x}\} \subseteq \mathbb{D}(Y)^{\otimes}$. Let $b \in G(\breve{\mathbb{Q}}_p) \subseteq \mathrm{GL}(V^*)(\breve{\mathbb{Q}}_p)$ be the element corresponding to the Frobenius on $\mathbb{D}(Y)[\frac{1}{p}]$. Under such an isomorphism, the Frobenius corresponds to an element $b = b_x \in G(\breve{\mathbb{Q}}_p) \subseteq \mathrm{GL}(V^*)(\breve{\mathbb{Q}}_p)$. In particular, we can apply the result of Section 4.4 and form the objects $\mathcal{H}_Y^{G,\circ} \subseteq \mathcal{H}_Y$ and $\mathbf{Aut}_G(\tilde{Y})$. We then consider the Dieudonné-Lie $\breve{\mathbb{Z}}_p$-algebra $\mathfrak{a}^+ = \mathbb{D}(\mathcal{H}_Y^{G,\circ})$ and $\mathfrak{a} = \mathfrak{a}^+ \otimes \breve{\mathbb{Q}}_p$.

We have seen in Section 4.4 that $\mathfrak{a} \simeq \mathrm{Lie}\, U_{\nu_b}$ and that $\mathbf{Aut}_G(\tilde{Y})^\circ \simeq \tilde{\Pi}(\mathfrak{a})$.

Hamacher and Kim define an Igusa variety $\mathrm{Ig}_{\mathrm{CS}} \to C^{\mathrm{perf}}_{\mathsf{G},[\![b]\!]}$ by pulling back $\mathrm{Ig}_{\mathrm{CS},\lambda} \to C^{\mathrm{perf}}_{(Y,\lambda)}$ along $C^{\mathrm{perf}}_{\mathsf{G},[\![b]\!]} \to C^{\mathrm{perf}}_{(Y,\lambda)}$, and then taking the closed subset where the universal isomorphism

$$X \simeq Y_{C_{(Y,\lambda)}}$$

preserves the tensors on geometric points, see [38, Section 5.1 and Lemma 5.1.1]. They then prove that $\mathrm{Ig}_{\mathrm{CS}} \to C^{\mathrm{perf}}_{\mathsf{G},[\![b]\!]}$ is a pro-étale torsor for $\underline{\mathbf{Aut}_G(Y)(\overline{\mathbb{F}}_p)} \subseteq \underline{\mathbf{Aut}_\lambda(Y)(\overline{\mathbb{F}}_p)}$ and that $\mathrm{Ig}_{\mathrm{CS}}$ is stable under the action of $\mathbf{Aut}_G(\tilde{Y}) \subseteq \mathbf{Aut}(\tilde{Y})$. Using the invariance of the étale site under perfections, we get a pro-étale torsor $\mathrm{Ig}_{\mathrm{M}} \to C_{\mathsf{G},[\![b]\!]}$ sitting in a cartesian diagram

$$\begin{array}{ccc} \mathrm{Ig}_{\mathrm{CS}} & \longrightarrow & \mathrm{Ig}_{\mathrm{M}} \\ \big\downarrow & & \big\downarrow \\ C^{\mathrm{perf}}_{\mathsf{G},[\![b]\!]} & \longrightarrow & C_{\mathsf{G},[\![b]\!]}, \end{array}$$

that fits in a commutative cube with (5.3.1). The map $\mathrm{Ig}_{\mathrm{CS}} \to C_{\mathsf{G},[\![b]\!]}$ is faithfully flat since $\mathrm{Ig}_{\mathrm{CS}} \to C^{\mathrm{perf}}_{\mathsf{G},[\![b]\!]}$ is a pro-étale cover and $C_{\mathsf{G},[\![b]\!]}$ is smooth. We want to prove that it is actually an fpqc torsor under $\mathbf{Aut}_G(Y)$. We first prove it at the level of infinitesimal neighbourhoods (defined as in [76, Tag 0AIX]).

Lemma 5.4.1. *If $y \in \mathrm{Ig}_{\mathrm{CS}}(\overline{\mathbb{F}}_p)$ is an $\overline{\mathbb{F}}_p$-point over $x \in C_{\mathsf{G},[\![b]\!]}(\overline{\mathbb{F}}_p)$, then $\mathrm{Ig}_{\mathrm{CS}}^{/y} \to C^{/x}_{\mathsf{G},[\![b]\!]}$ is a torsor in the fpqc topology under the group scheme $\mathbf{Aut}_G^\circ(Y)$.*

Proof. In [38, Section 5.2], Hamacher and Kim show that the formal completion at a point of $\mathrm{Ig}_{\mathrm{CS}}$ is isomorphic to $\mathbf{Aut}^{\circ}_{G}(\tilde{Y})$ and that the natural action of $\mathbf{Aut}^{\circ}_{G}(\tilde{Y})$ on $\mathrm{Ig}_{\mathrm{CS}}$ corresponds to the multiplication map under this isomorphism. Moreover, the morphism $\mathrm{Ig}_{\mathrm{CS}}^{/y} \to C^{/x}_{\mathsf{G},[\![b]\!]}$ corresponds to the restriction of the action map (constructed in [47, Theorem 4.3.1])

$$\mathbf{Aut}^{\circ}_{G}(\tilde{Y}) \times \mathrm{Sh}^{/x}_{\mathsf{G},U,[b]} \to \mathrm{Sh}^{/x}_{\mathsf{G},U,[b]}$$

to the closed point $x \in \mathrm{Sh}^{/x}_{\mathsf{G},U,[b]}$. It follows from [47, Theorem 5.1.3] that the scheme-theoretic image of this restriction

$$\mathbf{Aut}^{\circ}_{G}(\tilde{Y}) \to \mathrm{Sh}^{/x}_{\mathsf{G},U,[b]}$$

is $C^{/x}_{\mathsf{G},[\![b]\!]} \subseteq \mathrm{Sh}^{/x}_{\mathsf{G},U,[b]}$ and in fact this identifies $\mathbf{Aut}^{\circ}_{G}(\tilde{Y})$ with the perfection of $C^{/x}_{\mathsf{G},[\![b]\!]}$. Now we apply this to $(\mathsf{G},\mathsf{X}) = (\mathsf{G}_V, \mathsf{H}_V)$, where we already know that

$$\mathrm{Ig}_{\mathrm{CS},\lambda} \to \mathrm{Ig}_{\mathrm{M},\lambda}$$

is a torsor for $\mathbf{Aut}^{\circ}_{\lambda}(Y)$. Since the map $\mathrm{Ig}_{\mathrm{M},\lambda} \to C_{(Y,\lambda)}$ is pro-étale, it follows that on formal neighborhoods we get an equivariant map

$$\mathbf{Aut}^{\circ}_{\lambda}(\tilde{Y}) \to C^{/x}_{(Y,\lambda)}$$

which is an $\mathbf{Aut}^{\circ}_{\lambda}(Y)$-torsor in the fpqc topology. Because $\mathbf{Aut}^{\circ}_{G}(Y)$ is the intersection of $\mathbf{Aut}^{\circ}_{\lambda}(Y)$ with $\mathbf{Aut}_{G}(\tilde{Y})$, it follows that $\mathbf{Aut}^{\circ}_{G}(Y) \subseteq \mathbf{Aut}^{\circ}_{G}(\tilde{Y})$ is the stabiliser of $x \in C^{/x}_{\mathsf{G},[\![b]\!]}$. Therefore, the natural map

$$\mathbf{Aut}^{\circ}_{G}(\tilde{Y}) \to C^{/x}_{\mathsf{G},[\![b]\!]}$$

is a quasi-torsor for $\mathbf{Aut}^{\circ}_{G}(Y)$. It is also an fpqc cover, because $\mathrm{Ig}_{\mathrm{CS}} \to C_{\mathsf{G},[\![b]\!]}$ is, and thus it is a torsor for $\mathbf{Aut}^{\circ}_{G}(Y)$ in the fpqc topology. □

Proposition 5.4.2. *The map* $\mathrm{Ig}_{\mathrm{CS}} \to C_{\mathsf{G},[\![b]\!]}$ *is an fpqc-torsor under* $\mathbf{Aut}_{G}(Y)$.

Proof. Since $\mathrm{Ig}_{\mathrm{CS}} \to C_{\mathsf{G},[\![b]\!]}$ is faithfully flat, it suffices to prove that it is a quasi-torsor under $\mathbf{Aut}_{G}(Y)$. In other words, we want to show that the action map

$$\mathbf{Aut}_{G}(Y) \times \mathrm{Ig}_{\mathrm{CS}} \to \mathrm{Ig}_{\mathrm{CS}} \times_{C_{\mathsf{G},[\![b]\!]}} \mathrm{Ig}_{\mathrm{CS}} \tag{5.4.1}$$

is an isomorphism. This map is clearly a homeomorphism because $\mathrm{Ig}_{\mathrm{M}} \to C_{\mathsf{G},[\![b]\!]}$ is an $\underline{\mathbf{Aut}_{G}(Y)(\overline{\mathbb{F}}_p)}$-torsor and both $\mathrm{Ig}_{\mathrm{CS}} \to \mathrm{Ig}_{\mathrm{M}}$ and $\mathbf{Aut}_{G}(Y) \to \underline{\mathbf{Aut}_{G}(Y)(\overline{\mathbb{F}}_p)}$ are universal homeomorphisms.

It follows from [38, Lemma 5.2.3] that when Z is either the source or the target of (5.4.1) and $z \in Z(\overline{\mathbb{F}}_p)$ is an $\overline{\mathbb{F}}_p$-point over x, then $Z^{/z}$ is pro-represented by the formal spectrum of the I-adic completion of $\mathcal{O}_{Z,x}$, where I is the maximal ideal of $\mathcal{O}_{C_{\mathsf{G},[\![b]\!]},x}$. Moreover, they prove that $\mathcal{O}_{Z,x} \to \mathcal{O}^{\wedge,I}_{Z,x}$ is faithfully flat.

It follows from Lemma 5.4.1 that the action map (5.4.1) is flat at all closed points and therefore it is flat by [76, Lemma 00HT.(7)]. We moreover know that the action map is a closed immersion because this is true in the Siegel case, so the action map is a surjective flat closed immersion and therefore an isomorphism (see [76, Tag 04PW]). □

5.4.3. *Deformation theory of torsors.* If we let the notation be as above then Chai and Oort prove that the deformation theory of $C_{(Y,\lambda)}$ is completely determined by the $\mathbf{Aut}_\lambda(Y)$-torsor $\mathrm{Ig}_{\mathrm{CS},\lambda} \to C_{(Y,\lambda)}$. Let $\mathbf{Def}_{\mathrm{sus}}(Y,\lambda)$ denote the deformation space (considered as a functor on the category of Artin local rings over $\overline{\mathbb{F}}_p$ with residue field $\overline{\mathbb{F}}_p$) of the trivial $\mathbf{Aut}_\lambda(Y)$-torsor over $\overline{\mathbb{F}}_p$. Then by [14, Lemma 3.6, Theorem 4.3], the induced map

$$C_{(Y,\lambda)}^{/x} \to \mathbf{Def}_{\mathrm{sus}}(Y,\lambda)$$

is an isomorphism of formal schemes. They prove this by showing that $\mathbf{Def}_{\mathrm{sus}}(Y,\lambda)$ is formally smooth of the same dimension as $C_{(Y,\lambda)}^{/x}$. It is possible to similarly show that $C_{\mathsf{G},[\![b]\!]}^{/x}$ can be identified with the deformation space of the trivial $\mathbf{Aut}_G(Y)$-torsor, but this result does not enter into our proof.

5.4.4. If $Y/\overline{\mathbb{F}}_p$ is a completely slope divisible p-divisible group, then one can prove that the deformation space $\mathbf{Def}_{\mathrm{sus}}(Y)$ of the trivial $\mathbf{Aut}(Y)$-torsor admits a closed immersion to the deformation space $\mathbf{Def}(Y)$ of Y, see [14, Lemma 3.6, Theorem 4.3]. Its image is identified with the subspace of deformations of Y that are fpqc locally isomorphic to the constant deformation of Y. It follows from [47, Section 5] that there is an action of $\mathbf{Aut}(\tilde{Y})^\circ$ on $\mathbf{Def}_{\mathrm{sus}}(Y)$ and an equivariant map $\mathbf{Aut}(\tilde{Y})^\circ \to \mathbf{Def}_{\mathrm{sus}}(Y)$ which is an $\mathbf{Aut}(Y)^\circ$-torsor in the fpqc topology.

Remark 5.4.5. If $Y = Y_1 \oplus Y_2$ has two slopes, then $\mathbf{Aut}(\tilde{Y})^\circ$ is isomorphic to $\widetilde{\mathcal{H}}_{Y_1,Y_2}$ and $\mathbf{Aut}(Y)^\circ \simeq T_p\mathcal{H}_{Y_1,Y_2}$ so that $\mathbf{Def}_{\mathrm{sus}}(Y) \simeq \mathcal{H}_{Y_1,Y_2}$. This gives $\mathbf{Def}_{\mathrm{sus}}(Y)$ the structure of a p-divisible formal group. When Y is ordinary, then $\mathbf{Def}_{\mathrm{sus}}(Y) = \mathbf{Def}(Y)$ and the formal group structure on $\mathbf{Def}(Y)$ is the one coming from the classical Serre–Tate coordinates, see [42, Section 4].

5.5. **Strongly Tate-linear subspaces.** Let $Y/\overline{\mathbb{F}}_p$ be a completely slope divisible p-divisible group and let X be the universal p-divisible group over the sustained deformation space $\mathbf{Def}_{\mathrm{sus}}(Y)$. If $Z \subseteq \mathbf{Def}_{\mathrm{sus}}(Y)$ is a formally smooth closed subscheme, then the monodromy group[6] $G(\mathcal{M}_Z)$ of the isocrystal $\mathcal{M} = \mathbb{D}(X)[\frac{1}{p}]$ restricted to Z, with respect to the closed point of Z, has a natural inclusion

$$\mathrm{Lie}\, G(\mathcal{M}_Z) \subseteq \mathbb{D}(\mathcal{H}_Y^\circ)[\tfrac{1}{p}] \subseteq \mathbb{D}(\mathcal{H}_Y)[\tfrac{1}{p}] = \mathrm{Lie}\,\mathrm{GL}(\mathbb{D}(Y))[\tfrac{1}{p}].$$

Since $\mathcal{M}$ has the structure of an F-isocrystal we get, by [20, Section 2.2], an isomorphism

$$G(\mathcal{M}_Z)^{(p)} \to G(\mathcal{M}_Z),$$

which induces an isomorphism

$$\mathrm{Lie}\, G(\mathcal{M}_Z)^{(p)} \to \mathrm{Lie}\, G(\mathcal{M}_Z)$$

compatible with the F-structure on $\mathbb{D}(\mathcal{H}_Y^\circ)[\frac{1}{p}]$. In particular

$$\mathrm{Lie}\, G(\mathcal{M}_Z) \subseteq \mathbb{D}(\mathcal{H}_Y^\circ)[\tfrac{1}{p}] =: \mathfrak{a}$$

is an F-stable Lie subalgebra.

[6]The monodromy group is taken with respect to the closed point of Z. We will remove in this section the choice of the point in the notation.

5.5.1. We have seen in Section 5.4.4 that there is an action of $\mathbf{Aut}(\tilde{Y})^\circ$ on $\mathbf{Def}_{\mathrm{sus}}(Y)$ together with an equivariant map

$$\mathbf{Aut}(\tilde{Y})^\circ \to \mathbf{Def}_{\mathrm{sus}}(Y)$$

which is an $\mathbf{Aut}(Y)^\circ$-torsor in the fpqc topology. If we consider $\mathfrak{a}^+ = \mathbb{D}(\mathcal{H}_Y^\circ)$ as a nilpotent Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra, then it is completely slope divisible by Lemma 4.1.2. Nonetheless, it is generally *not* integrable, so we will often consider $p^2\mathfrak{a}^+$ instead which is plain by Lemma 4.2.10.

Lemma 5.5.2. *The exponential isomorphism $E\colon \tilde{\mathcal{H}}_Y^\circ \xrightarrow{\sim} \mathbf{Aut}(\tilde{Y})^\circ$ identifies $\Pi(p^2\mathfrak{a}^+)$ with the profinite normal subgroup scheme*

$$\mathbf{Aut}(Y)^\circ_{p^2} \subseteq \mathbf{Aut}(Y)^\circ,$$

of those automorphisms which are trivial when restricted to $Y[p^2]$. In particular, we get a short exact sequence of fpqc sheaves

$$1 \to \Pi(p^2\mathfrak{a}^+) \to \mathbf{Aut}(Y)^\circ \to H \to 1,$$

with $H \subseteq \mathbf{Aut}(Y[p^2])$ a finite group scheme over $\overline{\mathbb{F}}_p$.

Proof. The exponential restricts to a morphism

$$p^2 T_p\mathcal{H}_Y^\circ \to \mathbf{Aut}(Y)^\circ_{p^2},$$

since $\frac{p^{2n}}{n!} \in p^2\mathbb{Z}_p$ for every $n \geq 1$. Similarly, the logarithm induces a morphism

$$\mathbf{Aut}(Y)^\circ_{p^2} \to p^2 T_p\mathcal{H}_Y^\circ.$$

These morphisms are mutual inverses.

For the second part, noting that $\mathbf{Aut}([Y[p^2])$ is an affine group scheme[7], it follows from [78, second theorem on page 144] that there exists an affine closed subgroup $H \subseteq \mathbf{Aut}(Y[p^2])$ such that $\mathbf{Aut}(Y) \to \mathbf{Aut}(Y[p^2])$ factors through H and $\mathbf{Aut}(Y) \to H$ is faithfully flat. It follows from the proof of [7, Lemma 4.1.5] that H is a finite group scheme. This completes the proof. □

Lemma 5.5.3. *There is a unique morphism $Z(p^2\mathfrak{a}^+) \to \mathbf{Def}_{\mathrm{sus}}(Y)$ making the following diagram commute*

$$\begin{array}{ccc} \tilde{\Pi}(\mathfrak{a}) & = & \tilde{\Pi}(\mathfrak{a}) \\ \downarrow & & \downarrow \\ Z(p^2\mathfrak{a}^+) & \dashrightarrow & \mathbf{Def}_{\mathrm{sus}}(Y). \end{array}$$

Moreover, it is finite, faithfully flat and $\tilde{\Pi}(\mathfrak{a})$-equivariant.

Proof. The free action of $\mathbf{Aut}(Y)^\circ$ on $\tilde{\Pi}(\mathfrak{a})$ induces a free action of H on $Z(p^2\mathfrak{a}^+)$ and we consider the fppf quotient $Z(p^2\mathfrak{a}^+)/H$. Since the H-action preserves the presentation of $Z(p^2\mathfrak{a}^+)$ as the colimit of the schemes $Z^n(p^2\mathfrak{a}^+)$, it follows that $Z(p^2\mathfrak{a}^+)/H$ is representable by a formal scheme. The map $\tilde{\Pi}(\mathfrak{a}) \to Z(p^2\mathfrak{a}^+)/H$ is faithfully flat and a quasi-torsor for $\mathbf{Aut}(Y)^\circ$, it is thus a torsor for $\mathbf{Aut}(Y)^\circ$. It follows from the proof of Lemma 4.3.11 that there is an isomorphism $Z(p^2\mathfrak{a}^+)/H \to \mathbf{Def}_{\mathrm{sus}}(Y)$, proving the lemma. □

[7]The automorphism group of an affine finite k-scheme $\operatorname{Spec} A$ is a closed subgroup of $\mathrm{GL}(A)$, hence affine. It follows that the automorphism group of a finite group scheme is affine.

5.5.4. For a sub-F-isocrystal $\mathfrak{b} \subseteq \mathfrak{a}$ stable under the Lie bracket we define $\mathfrak{b}^+ = \mathfrak{b} \cap \mathfrak{a}^+$. This is a completely slope divisible and nilpotent Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra, see Lemma 2.3.15. It follows from Lemma 4.2.10 that $p^2\mathfrak{b}^+$ is plain. We can then consider the formal homogenous space $Z(p^2\mathfrak{b}^+)$, which comes equipped with a natural monomorphism $Z(p^2\mathfrak{b}^+) \to Z(p^2\mathfrak{a}^+)$ which must be a closed immersion by Lemma 2.1.2.

Let $\mathbf{Aut}_{\mathfrak{b}^+}(Y)^\circ$ be the intersection of $\mathbf{Aut}(Y)^\circ$ with $\tilde{\Pi}(\mathfrak{b})$. This is an affine group scheme containing $\Pi(\mathfrak{b}^+)$ as a closed subgroup; in fact if H is an in the statement of Lemma 5.5.2, then $\Pi(\mathfrak{b}^+) \subset \mathbf{Aut}_{\mathfrak{b}^+}(Y)^\circ$ is the kernel of the natural map $\mathbf{Aut}_{\mathfrak{b}^+}(Y)^\circ \to H$. If we let H_2 be the scheme-theoretic image of this natural map, then it follows from [78, second theorem on page 144] that $\mathbf{Aut}_{\mathfrak{b}^+}(Y)^\circ \to H_2$ is faithfully flat. We deduce from this and Lemma 4.3.12 that there is a natural action of H_2 on $Z(p^2\mathfrak{b}^+)$. Moreover, the natural map $Z(p^2\mathfrak{b}^+) \to Z(p^2\mathfrak{a}^+) \to Z(\mathfrak{a}^+)$ is H-invariant inducing a monomorphism $Z(p^2\mathfrak{b}^+)/H \to Z(\mathfrak{a}^+)$, which is a closed immersion by Lemma 2.1.2. We define $Z(\mathfrak{b}^+)$ to be the quotient $Z(p^2\mathfrak{b}^+)/H$. We are now in the position to state the following conjecture.

Conjecture 5.5.5. *Let $Z \hookrightarrow \mathbf{Def}_{\mathrm{sus}}(Y)$ be a closed immersion. For each F-stable Lie subalgebra $\mathfrak{b} \subseteq \mathfrak{a}$ there is an inclusion $Z \subseteq Z(\mathfrak{b}^+)$ if and only if* $\operatorname{Lie} G(\mathcal{M}_Z) \subseteq \mathfrak{b}$. *In particular* $\operatorname{Lie} G(\mathcal{M}_{Z(\mathfrak{b}^+)}) = \mathfrak{b}$.

Example 5.5.6. Let the notation be as in Section 5.2 and consider a central leaf $C_{\mathsf{G},[\![b]\!]}$ lying in a $\mathbb{Q}$-non-basic Newton stratum (see Definition 8.3.1 below). Take a point $x \in C_{\mathsf{G},[\![b]\!]}$ with corresponding completely slope divisible p-divisible group Y, and let $\mathfrak{b}^+ = \mathbb{D}(\mathcal{H}_Y^{G,\circ})$. It follows from the discussions above that $Z(\mathfrak{b}^+) \subseteq \mathbf{Def}_{\mathrm{sus}}(Y)$ can be identified with $C_{\mathsf{G},[\![b]\!]}^{/x} \subseteq \mathbf{Def}_{\mathrm{sus}}(Y)$ (both are the scheme-theoretic image of $\tilde{\Pi}(\mathfrak{b}) \to \mathbf{Def}_{\mathrm{sus}}(Y)$). In this case, the unipotent radical of the monodromy group

$$\operatorname{Mon}(C_{\mathsf{G},[\![b]\!]}, \mathcal{M})$$

is isomorphic to U_{ν_b} by [42, Corollary 3.3.5][8], which uses [21, 22], and then Theorem 3.4.4 allows us to conclude. In particular, we see that the monodromy group of $\mathcal{M}$ over $Z(\mathfrak{b}^+)$ has Lie algebra $\mathfrak{b}$.

As a special case of this if Y has height h and dimension d, then $\mathbf{Def}_{\mathrm{sus}}(Y)$ can be realised as the complete local ring of a central leaf in a PEL type unitary Shimura variety of signature $(h-d, d)$ associated to an imaginary quadratic field E in which p splits. In particular, we know that the monodromy group of $\mathcal{M}/\mathbf{Def}_{\mathrm{sus}}(Y)$ is isomorphic to the unipotent group corresponding to the nilpotent Lie algebra $\mathbb{D}(\mathcal{H}_Y^\circ)[\frac{1}{p}] = \mathfrak{a}$.

Construction 5.5.7 ($U(\mathfrak{b})$)**.** There is a $\breve{\mathbb{Z}}_p$-algebra structure on $\mathbb{D}(\mathcal{H}_Y)$ and $\mathfrak{a}^+ := \mathbb{D}(\mathcal{H}_Y^\circ) \subseteq \mathbb{D}(\mathcal{H}_Y)$ is an algebra ideal (and Lie subalgebra). In particular $1 + p^2\mathfrak{a}^+$ is a subgroup of its group of units which defines a unipotent algebraic group over $\breve{\mathbb{Z}}_p$. After inverting p, it follows from that BCH formula that we may identify $1 + \mathfrak{a}$ with the exponential of the Lie $\breve{\mathbb{Q}}_p$-algebra $\mathfrak{a}$. Given $\mathfrak{b} \subseteq \mathfrak{a}$, we therefore get a subgroup $1 + \mathfrak{b}$ defined by the exponential of the Lie $\breve{\mathbb{Q}}_p$-algebra $\mathfrak{b}$. We define $U(\mathfrak{b}^+)$ to be the intersection of the subgroups $1 + p^2\mathfrak{a}^+$ and $1 + \mathfrak{b}$, which we can identify with $1 + p^2\mathfrak{b}^+$ where $\mathfrak{b}^+ = \mathfrak{a}^+ \cap \mathfrak{b}$. This defines a unipotent group $U(\mathfrak{b}^+)$ over $\breve{\mathbb{Z}}_p$. We write $U(\mathfrak{b})$ for the base change to $\breve{\mathbb{Q}}_p$.

[8] The statement of [42, Corollary 3.3.5] contains the assumption that [42, Hypothesis 2.3.1] holds. This is true for us because K_p is hyperspecial, see [42, Lemma 2.3.2].

6. Local monodromy of strongly Tate-linear subvarieties

6.1. **Introduction.** In this section we will prove half of Conjecture 5.5.5. Let $Y/\overline{\mathbb{F}}_p$ be a completely slope divisible p-divisible group and let X be the universal p-divisible group over the sustained deformation space $\mathbf{Def}_{\mathrm{sus}}(Y)$. Let $\mathfrak{a}^+ = \mathbb{D}(\mathcal{H}_Y^\circ)$ be the Dieudonné–Lie algebra associated to the internal-Hom p-divisible group of Y and let $\mathfrak{b} \subseteq \mathfrak{a}$ be an F-stable Lie subalgebra with associated strongly Tate-linear subspace $Z(\mathfrak{b}^+) \subseteq \mathbf{Def}_{\mathrm{sus}}(Y)$. Write $\mathcal{M} = \mathbb{D}(X)[\frac{1}{p}]$ for the isocrystal over $\mathbf{Def}_{\mathrm{sus}}(Y)$ coming from the Dieudonné module of X.

Theorem 6.1.1. *There is a natural closed immersion $G(\mathcal{M}_{Z(\mathfrak{b}^+)}) \hookrightarrow U(\mathfrak{b})$.*

In the proof we will make use of the Cartier–Witt stacks of Drinfeld [29] and Bhatt–Lurie [3] associated to quasi-syntomic schemes of characteristic p. Given such a scheme X, there is a p-adic formal stack X^{Δ}, the *prismatisation* of X, such that coherent crystals on X are the same as coherent sheaves on X^{Δ}. In Section 6.2 we will give a more detailed overview of this construction and its properties.

We will now give a sketch of the proof. Consider the $\mathbf{Aut}(Y)$-torsor

$$\mathbf{Isom}(X, Y_{\mathbf{Def}_{\mathrm{sus}}(Y)}) \to \mathbf{Def}_{\mathrm{sus}}(Y) \tag{6.1.1}$$

over $\mathbf{Def}_{\mathrm{sus}}(Y)$. The locally free crystal $\mathcal{M}^+ = \mathbb{D}(X)$ defines a vector bundle $\mathcal{V}^+$ over the prismatisation[9] $\mathbf{Def}_{\mathrm{sus}}(Y)^{\Delta}$. This vector bundle has an associated frame bundle which we will write suggestively as

$$\mathbf{Isom}(\mathcal{V}^+, \mathbb{D}(Y)_{\mathbf{Def}_{\mathrm{sus}}(Y)^{\Delta}}). \tag{6.1.2}$$

If we apply the prismatisation functor to the map (6.1.1), then we get a morphism

$$\mathbf{Isom}(X, Y_{\mathbf{Def}_{\mathrm{sus}}(Y)})^{\Delta} \to \mathbf{Def}_{\mathrm{sus}}(Y)^{\Delta},$$

which will be a torsor for the p-adic formal group $\mathbf{Aut}(Y)^{\Delta}$ (this will follow from Lemma 6.3.2). Dieudonné-theory gives us a homomorphism of group schemes over Spf $\mathbb{Z}_p$

$$\mathbf{Aut}(Y)^{\Delta} \to \mathbf{Aut}(\mathbb{D}(Y)) \tag{6.1.3}$$

and a morphism

$$\mathbf{Isom}(X, Y_{\mathbf{Def}_{\mathrm{sus}}(Y)})^{\Delta} \to \mathbf{Isom}(\mathcal{V}^+, \mathbb{D}(Y)_{\mathbf{Def}_{\mathrm{sus}}(Y)^{\Delta}}),$$

which is $\mathbf{Aut}(Y)^{\Delta}$-equivariant via the homomorphism (6.1.3). The right hand side roughly speaking parametrises all isomorphisms between $\mathcal{V}^+$ and $\mathbb{D}(Y)_{\mathbf{Def}_{\mathrm{sus}}(Y)^{\Delta}}$, while the left hand side parametrises those isomorphisms that are compatible with the F-structures.

So how does this help us? After pulling back via the finite flat cover $Z = Z(p^2\mathfrak{b}^+) \to Z(\mathfrak{b}^+)$, the torsor (6.1.1) has a reduction to a $\Pi(p^2\mathfrak{b}^+)$-torsor by construction. Feeding this fact into the prismatisation machinery will give us a reduction of the torsor (6.1.2) to a $U(\mathfrak{b}^+)$-torsor. If we apply the Tannakian perspective on torsors and invert p, then this will exactly give us a closed immersion

$$G(\mathcal{M}_Z) \to U(\mathfrak{b}),$$

which is exactly what we want to prove once we identify $G(\mathcal{M}_Z) = G(\mathcal{M}_{Z(\mathfrak{b}^+)})$ using Proposition 3.3.7.

[9]We pretend for now that our formal schemes are actually schemes, in the actual proof there is an additional algebraisation step.

6.2. **Cartier–Witt stacks.** Let us briefly recall the main properties of the *Cartier–Witt stacks* from [29], [3]. We will deal only with *quasi-syntomic schemes* X over $\mathbb{F}_p$, as in Definition 2.3.4.

6.2.1. Write $\mathrm{Nilp}_{\mathbb{Z}_p}^{\mathrm{op}} \subseteq \mathrm{Alg}_{\mathbb{Z}_p}^{\mathrm{op}}$ for the full subcategory of p-nilpotent algebras with the fpqc topology. A *p-adic formal stack* is a groupoid valued functor $\mathcal{F}$ on $\mathrm{Nilp}_{\mathbb{Z}_p}^{\mathrm{op}}$ whose diagonal is representable in formal algebraic spaces and which admits an fpqc cover $\mathcal{X} \to \mathcal{F}$, where $\mathcal{X}$ is a p-adic formal algebraic space over $\mathrm{Spf}\,\mathbb{Z}_p$ (cf. [76, Tag 0AIM]).

Drinfeld and Bhatt–Lurie define a prismatisation functor

$$X \mapsto X^{\Delta}$$

which goes from the category of quasi-syntomic $\mathbb{F}_p$-schemes to the category of p-adic formal stacks endowed with an endomorphism $F : X^{\Delta} \to X^{\Delta}$, lifting the Frobenius on the special fibre[10].

6.2.2. For every quasi-syntomic scheme X, it follows from [3, Proposition 8.15] that there is an equivalence between the category of crystals in quasi-coherent $\mathcal{O}$-modules on the absolute prismatic site of X (or the absolute crystalline site by [4, Example 4.7]) and quasi-coherent $\mathcal{O}$-modules on the Zariski site of X^{Δ}.

There are a few important properties of this functor that we will use.

- If $X = \mathrm{Spec}\,R$ is a semiperfect quasi-syntomic scheme, then X^{Δ} is simply $\mathrm{Spf}\,A_{\mathrm{cris}}(R)$, the formal spectrum of Fontaine's ring of crystalline periods (this is [3, Lemma 6.1]). For example $(\mathrm{Spec}\,\mathbb{F}_p)^{\Delta} = \mathrm{Spf}\,\mathbb{Z}_p$.
- If $f : X \to Y$ is a quasi-syntomic cover, see Definition 2.3.4, then $f : X^{\Delta} \to Y^{\Delta}$ is an fpqc cover (this is [3, Proposition 7.5]). For example, this means that $X^{\Delta} \to (\mathrm{Spec}\,\mathbb{F}_p)^{\Delta} = \mathrm{Spf}\,\mathbb{Z}_p$ is automatically flat if X is qrsp.
- The functor commutes with products and with fibre products in the case that the structure maps are flat and quasi-syntomic, by [3, Proposition 7.5, Remark 8.9].

6.3. **Algebraisation.** Let $Z(\mathfrak{b}^+) \subseteq \mathbf{Def}_{\mathrm{sus}}(Y)$ be a strongly Tate-linear formal closed subscheme of $\mathbf{Def}_{\mathrm{sus}}(Y)$ corresponding to an F-stable Lie subalgebra $\mathfrak{b} \subset \mathfrak{a}$, and let $Z = Z(p^2\mathfrak{b}^+) \to Z(\mathfrak{b}^+)$. The formal scheme Z is equal to $\mathrm{Spf}\,A$ for a complete Noetherian local ring A. It follows from [23, Proposition 2.4.8] that the category of Dieudonné isocrystals over $\mathrm{Spf}\,A$ is equivalent to the category of Dieudonné isocrystals over $\mathrm{Spec}\,A$. Moreover, we have that the $\Pi(\mathfrak{b}^+)$-torsor P_Z over $Z = \mathrm{Spf}\,A$ comes from a $\Pi(\mathfrak{b}^+)$-torsor P_Z^{alg} over $\mathrm{Spec}\,A$. Indeed P is an inverse limit of torsors under finite flat group schemes over Z, and those all algebraise because finite modules do functorially.

Notation 6.3.1. In Section 6 we will treat Z as the affine scheme $\mathrm{Spec}\,A$ rather than the formal scheme $\mathrm{Spf}\,A$ and we will simply write $P_Z \to Z$ for the algebraic torsor P_Z^{alg} defined above. The same applies to $\mathbf{Def}_{\mathrm{sus}}(Y)$.

Let $\mathcal{M}^+$ be the F-crystal over $\mathbf{Def}_{\mathrm{sus}}(Y)$ attached to the universal p-divisible group. We write $\mathcal{M}$ for the induced F-isocrystal and $\mathcal{M}_Z$ the restriction of $\mathcal{M}$ to Z.

Proof of Theorem 6.1.1. Let $P \to Z$ be the restriction of the universal $\mathbf{Aut}(Y)$-torsor and let $P_Z \to Z$ be the reduction of P to a $\Pi(p^2\mathfrak{b}^+)$-torsor. The basic idea of the proof is to use descent of isocrystals along $P_Z \to Z$ to describe $\mathcal{M}_{P_Z}$ as a constant isocrystal equipped with a descent datum (or equivalently a $\Pi(p^2\mathfrak{b}^+)$-equivariant structure). However it seems quite hard to compare

[10] They also define a derived version of this functor, which we will not use in this text.

the group scheme $\Pi(p^2\mathfrak{b}^+)$ over $\overline{\mathbb{F}}_p$ with the monodromy group of $\mathcal{M}$, which is an algebraic group over $\breve{\mathbb{Q}}_p$. This is where the Cartier–Witt stacks of [29],[3] come in.

The Dieudonné module $\mathbb{D}(Y)$ of Y is a trivial vector bundle on $\overline{\mathbb{F}}_p^{\mathbb{\Delta}} = \mathrm{Spf}(\breve{\mathbb{Z}}_p)$ endowed with a Frobenius. We denote by $\mathrm{GL}(\mathbb{D}(Y))$ the p-adic formal group over $\mathrm{Spf}(\breve{\mathbb{Z}}_p)$ of $\breve{\mathbb{Z}}_p$-linear automorphisms of $\mathbb{D}(Y)$ (thus forgetting the F-structure). Let $U(\mathfrak{b}^+) \subseteq \mathrm{GL}(\mathbb{D}(Y))$ denote the inclusion of the p-adic completion of the unipotent group $U(\mathfrak{b}^+)$.

By the formalism of Cartier–Witt stacks, the crystal $\mathcal{M}^+$ corresponds to a vector bundle $\mathcal{V}^+$ on $Z^{\mathbb{\Delta}}$. In turn, this defines a formal $\mathrm{GL}(\mathbb{D}(Y))$-torsor

$$\mathbf{Isom}(\mathcal{V}^+, \mathbb{D}(Y)_{Z^{\mathbb{\Delta}}}) \to Z^{\mathbb{\Delta}}$$

over $Z^{\mathbb{\Delta}}$ which we denote by $\mathfrak{Q} \to Z^{\mathbb{\Delta}}$. On the other hand, $P_Z \to Z$ induces a $\Pi(p^2\mathfrak{b}^+)^{\mathbb{\Delta}}$-torsor $P_Z^{\mathbb{\Delta}} \to Z^{\mathbb{\Delta}}$ of p-adic formal stacks by the following lemma.

Lemma 6.3.2. *Let T be a torsor over a quasi-syntomic scheme S over $\mathbb{F}_p$ under a qrsp group scheme G. The prismatisation $G^{\mathbb{\Delta}}$ of G is a formal group scheme and $T^{\mathbb{\Delta}} \to S^{\mathbb{\Delta}}$ is a $G^{\mathbb{\Delta}}$-torsor of formal stacks.*

Proof. We will make use of the properties of prismatisation outlined in Section 6.2.2. Since G is qrsp, its prismatisation is a p-adic formal scheme. In addition, since prismatisation of $\mathbb{F}_p$-schemes commutes with products, it follows that $G^{\mathbb{\Delta}}$ is a formal group scheme. The fact that prismatisation sends quasi-syntomic covers to fpqc covers and commutes with fibre products when the structure maps are quasi-syntomic covers tells us that $T^{\mathbb{\Delta}} \to S^{\mathbb{\Delta}}$ is a torsor for $G^{\mathbb{\Delta}}$. □

6.3.3. We may apply Lemma 6.3.2 in our situation since the group scheme $\Pi(p^2\mathfrak{b}^+)$ over $\overline{\mathbb{F}}_p$ is qrsp by the discussion in Section 2.3.8. Write $\Pi(p^2\mathfrak{b}^+) = \mathrm{Spec}\, R$ and consider the tautological element $g_{\mathrm{univ}} \in \Pi(p^2\mathfrak{b}^+)(R)$. This element corresponds to an automorphism

$$g_{\mathrm{univ}} : Y_R \to Y_R,$$

and this induces an automorphism of Dieudonné modules

$$\mathbb{D}(g_{\mathrm{univ}}) : \mathbb{D}(Y) \otimes_{\breve{\mathbb{Z}}_p} A_{\mathrm{cris}}(R) \to \mathbb{D}(Y) \otimes_{\breve{\mathbb{Z}}_p} A_{\mathrm{cris}}(R).$$

This corresponds precisely to a $\mathrm{Spf}\, A_{\mathrm{cris}}(R) = \Pi(p^2\mathfrak{b}^+)^{\mathbb{\Delta}}$-point of $\mathrm{GL}(\mathbb{D}(Y))$, in other words, it corresponds to a map

$$\rho : \Pi(p^2\mathfrak{b}^+)^{\mathbb{\Delta}} \to \mathrm{GL}(\mathbb{D}(Y)).$$

Lemma 6.3.4. *The image of ρ lands in the closed subgroup $U(\mathfrak{b}^+) \subseteq \mathrm{GL}(\mathbb{D}(Y))$. Moreover, the morphism ρ is a group homomorphism.*

Proof. The definition of $\Pi(p^2\mathfrak{b}^+)$ tells us that g_{univ} is of the form $1 + p^2 f$ where $f \in T_p\mathbb{X}(\mathfrak{b}^+)$. Therefore $\mathbb{D}(g_{\mathrm{univ}})$ has the form $1 + p^2\mathbb{D}(f)$, and therefore lies in

$$1 + p^2\mathfrak{b}^+ \otimes_{\breve{\mathbb{Z}}_p} \mathrm{A}_{\mathrm{cris}}(R) \subseteq \mathrm{End}(\mathbb{D}(Y) \otimes_{\breve{\mathbb{Z}}_p} A_{\mathrm{cris}}(R)),$$

and therefore ρ factors through the unipotent group associated to $\mathfrak{b}^+$. The second claim of the lemma is that the following diagram commutes (where the vertical maps are the multiplication maps)

$$\begin{array}{ccc}
\Pi(p^2\mathfrak{b}^+)^{\mathbbl{\Delta}} \times \Pi(p^2\mathfrak{b}^+)^{\mathbbl{\Delta}} & \xrightarrow{\rho} & \mathrm{GL}(\mathbb{D}(Y)) \times \mathrm{GL}(\mathbb{D}(Y)) \\
\downarrow & & \downarrow \\
\Pi(p^2\mathfrak{b}^+)^{\mathbbl{\Delta}} & \longrightarrow & \mathrm{GL}(\mathbb{D}(Y)).
\end{array}$$

For $i = 1, 2$ let $p_{i,\mathrm{GL}} : \mathrm{GL}(\mathbb{D}(Y)) \times \mathrm{GL}(\mathbb{D}(Y)) \to \mathrm{GL}(\mathbb{D}(Y))$ and $p_{i,\Pi} : \Pi(p^2\mathfrak{b}^+)^{\mathbbl{\Delta}} \times \Pi(p^2\mathfrak{b}^+)^{\mathbbl{\Delta}} \to \Pi(p^2\mathfrak{b}^+)^{\mathbbl{\Delta}}$ be the projection maps. Using the Yoneda lemma, it suffices to show the equality

$$p_{1,\mathrm{GL}}^*\mathbb{D}(g_{\mathrm{univ}}) \circ p_{2,\mathrm{GL}}^*\mathbb{D}(g_{\mathrm{univ}}) = \mathbb{D}(p_{1,\Pi}^* g_{\mathrm{univ}} \circ p_{2,\Pi}^* g_{\mathrm{univ}})$$

as elements of

$$\mathrm{GL}(\mathbb{D}(Y))\left(\Pi(p^2\mathfrak{b}^+)^{\mathbbl{\Delta}} \times \Pi(p^2\mathfrak{b}^+)^{\mathbbl{\Delta}}\right) = \mathbf{Aut}(\mathbb{D}(Y) \otimes_{\breve{\mathbb{Z}}_p} \mathrm{A}_{\mathrm{cris}}(R \otimes_{\overline{\mathbb{F}}_p} R)).$$

But functoriality of Dieudonné-theory tells us that

$$\begin{aligned}
\mathbb{D}(p_{1,\Pi}^* g_{\mathrm{univ}} \circ p_{2,\Pi}^* g_{\mathrm{univ}}) &= \mathbb{D}(p_{1,\Pi}^* g_{\mathrm{univ}}) \circ \mathbb{D}(p_{2,\Pi}^* g_{\mathrm{univ}}) \\
&= p_{1,\mathrm{GL}}^*\mathbb{D}(g_{\mathrm{univ}}) \circ p_{2,\mathrm{GL}}^*\mathbb{D}(g_{\mathrm{univ}}).
\end{aligned}$$

□

6.3.5. There is an isomorphism

$$h_{\mathrm{univ}} : X_{P_Z} \to Y_{P_Z}$$

corresponding to the identity map $P_Z \to P_Z$. Applying the Dieudonné-theory functor we get an isomorphism

$$\mathcal{V}^+_{P_Z^{\mathbbl{\Delta}}} \to \mathbb{D}(Y)_{P_Z^{\mathbbl{\Delta}}},$$

which corresponds to a morphism

$$\sigma : P_Z^{\mathbbl{\Delta}} \to \mathfrak{Q}.$$

Lemma 6.3.6. *The map σ is $\Pi(p^2\mathfrak{b}^+)^{\mathbbl{\Delta}}$-equivariant, where $\Pi(p^2\mathfrak{b}^+)^{\mathbbl{\Delta}}$ acts on $\mathfrak{Q}$ via ρ.*

Proof. We are trying to show that the following diagram commutes

$$\begin{array}{ccc}
\Pi(p^2\mathfrak{b}^+)^{\mathbbl{\Delta}} \times P_Z^{\mathbbl{\Delta}} & \xrightarrow{(\rho,\sigma)} & \mathrm{GL}(\mathbb{D}(Y)) \times \mathfrak{Q} \\
\downarrow & & \downarrow \\
P_Z^{\mathbbl{\Delta}} & \xrightarrow{\sigma} & \mathfrak{Q},
\end{array}$$

where the vertical maps are given by the respective action maps. It suffices to prove that the diagram commutes on $\Pi(p^2\mathfrak{b}^+)^{\mathbbl{\Delta}} \times P_Z^{\mathbbl{\Delta}}$-points, which we will write as $\mathrm{Spf}\, A_{\mathrm{cris}}(R) \times \mathrm{Spf}\, A_{\mathrm{cris}}(S)$. The identity map $R \otimes S \to R \otimes S$ corresponds to $(g_{\mathrm{univ}}^{\mathbbl{\Delta}}, h_{\mathrm{univ}}^{\mathbbl{\Delta}})$ and (ρ, σ) corresponds to $(\mathbb{D}(g_{\mathrm{univ}}), \mathbb{D}(h_{\mathrm{univ}}))$. The map to $P_Z^{\mathbbl{\Delta}}$ corresponds to the composition $g_{\mathrm{univ}} \circ h_{\mathrm{univ}}$. The commutativity of the diagram is equivalent to the equality

$$\mathbb{D}(g_{\mathrm{univ}} \circ h_{\mathrm{univ}}) = \mathbb{D}(g_{\mathrm{univ}}) \circ \mathbb{D}(h_{\mathrm{univ}}),$$

which follows from functoriality of Dieudonné-theory. □

Since ρ factors through $U(\mathfrak{b}^+)$, we get a reduction of the $\mathrm{GL}\left(\mathbb{D}(Y)\right)$-torsor $\mathfrak{Q} \to Z^{\mathbb{\Delta}}$ to a $U(\mathfrak{b}^+)$-torsor $\mathfrak{R} \to Z^{\mathbb{\Delta}}$ sitting between $P_Z^{\mathbb{\Delta}}$ and $\mathfrak{Q}$. We can associate to this the symmetric tensor functor

$$\Psi : \mathrm{Rep}_{\breve{\mathbb{Z}}_p}(U(\mathfrak{b}^+)) \to \mathrm{Vect}(Z^{\mathbb{\Delta}})$$

which sends $V \in \mathrm{Rep}_{\breve{\mathbb{Z}}_p}(U(\mathfrak{b}^+))$ to

$$\mathfrak{R} \times^{U(\mathfrak{b}^+)} (V \otimes_{\breve{\mathbb{Z}}_p} Z^{\mathbb{\Delta}}).$$

The tautological representation $U(\mathfrak{b}^+) \hookrightarrow \mathrm{GL}\left(\mathbb{D}(Y)\right)$ is sent by Ψ to the vector bundle $\mathcal{V}^+$.

In order to pass to the generic fibre, we need the following result, which is a special case of [79, Proposition stated in Section 6.4].

Lemma 6.3.7. *Let $\mathfrak{G}$ be a smooth group scheme over $\breve{\mathbb{Z}}_p$ with generic fibre $\mathfrak{G}_\eta$. Then every representation $\rho : \mathfrak{G}_\eta \to \mathrm{GL}(V)$, where V is a finite dimensional $\breve{\mathbb{Q}}_p$-vector space, extends to a representation $\mathfrak{G} \to \mathrm{GL}(\Lambda)$ for some $\breve{\mathbb{Z}}_p$-lattice $\Lambda \subseteq V$.*

Applying the lemma and passing to isogeny categories, we get an exact tensor functor

$$\Psi_{\breve{\mathbb{Q}}_p} : \mathrm{Rep}_{\breve{\mathbb{Q}}_p}(U(\mathfrak{b})) \to \mathrm{Vect}(Z^{\mathbb{\Delta}})[\tfrac{1}{p}]$$

sending the defining representation of $U(\mathfrak{b})$ to $\mathcal{V}$. We can compose this with the natural inclusion

$$\mathrm{Vect}(Z^{\mathbb{\Delta}})[\tfrac{1}{p}] \hookrightarrow \mathrm{Isoc}(Z)$$

and apply Tannaka duality to get a morphism of group schemes

$$G(\mathcal{M}_Z) \to U(\mathfrak{b}).$$

This is a closed immersion because the constructed functor

$$\Psi_{\breve{\mathbb{Q}}_p} : \mathrm{Rep}_{\breve{\mathbb{Q}}_p}(U(\mathfrak{b})) \to \mathrm{Isoc}(Z)$$

between Tannakian categories commutes with the $\breve{\mathbb{Q}}_p$-linear fibre functor obtained by restricting the objects to the closed point of Z (see [26, Proposition 2.21.(b)]). By Proposition 3.3.7, there is a natural isomorphism

$$G(\mathcal{M}_{Z(\mathfrak{b}^+)}) \to G(\mathcal{M}_Z)$$

and thus we get a closed immersion $G(\mathcal{M}_{Z(\mathfrak{b}^+)}) \to U(\mathfrak{b})$ as desired. □

7. Rigidity

7.1. **Statement.** Let $(\mathfrak{a}^+, \varphi_{\mathfrak{a}^+}, [-,-])$ be a plain Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebra and let Q be an algebraic group over $\mathbb{Q}_p$ together with a strongly non-trivial action on $(\mathfrak{a}, \varphi_{\mathfrak{a}}, [-,-])$ (see Definition 4.5.1), and let $\Gamma \subseteq Q(\mathbb{Q}_p)$ be a compact open subgroup preserving $\mathfrak{a}^+$. Recall that for any F-stable Lie subalgebra $\mathfrak{b} \subseteq \mathfrak{a}$ there is a plain Dieudonné–Lie $\breve{\mathbb{Z}}_p$-subalgebra $\mathfrak{b}^+ \subseteq \mathfrak{a}^+$ defined by $\mathfrak{b}^+ = \mathfrak{b} \cap \mathfrak{a}^+$. In particular, such a $\mathfrak{b}$ defines a subspace $Z(\mathfrak{b}^+) \subseteq Z(\mathfrak{a}^+)$. The main goal of this section is to explain that the following theorem follows from [18, Theorem 5.1].

Theorem 7.1.1 (Rigidity)**.** *If $Z \subseteq Z(\mathfrak{a}^+)$ is a Γ-stable integral closed formal subscheme, then there is an F-stable Lie subalgebra $\mathfrak{b}_Z \subseteq \mathfrak{a}$ such that $Z = Z(\mathfrak{b}_Z^+)$.*

In other words, every subspace $Z \subseteq Z(\mathfrak{a}^+)$ that is stable under a strongly non-trivial action of a p-adic Lie group is strongly Tate-linear. Theorem 7.1.1 is essentially equivalent to [18, Theorem 5.1], although it is stated in a different language. In the next section, we will translate the language of Dieudonné–Lie $\breve{\mathbb{Z}}_p$-algebras into the language of Tate unipotent groups of [*ibid.*].

7.2. **Tate unipotent groups.** Equip $\Pi(\mathfrak{a}^+)$ with the filtration by normal subgroups $\mathrm{Fil}^\bullet \Pi(\mathfrak{a}^+)$ given by $\mathrm{Fil}^\lambda N = \Pi(\mathfrak{a}^+_{>-\lambda})$ for $\lambda \in (0,1]$. For all $i \geq 0$ there is an induced filtration by normal subgroups $\mathrm{Fil}^\bullet \Pi_i(\mathfrak{a}^+)$ of $\Pi_i(\mathfrak{a}^+)$ for each i. The family

$$\left\{\Pi_i(\mathfrak{a}^+), \mathrm{Fil}^\bullet \Pi_i(\mathfrak{a}^+)\right\}_{i\in\mathbb{Z}_{\geq 0}}$$

is a terraced Tate unipotent group over $\overline{\mathbb{F}}_p$ in the sense of [18, Definition 3.1]. In particular this means that $\Pi(\mathfrak{a}^+) = \varprojlim_i \Pi_i(\mathfrak{a}^+)$ is a Tate unipotent group in the sense of [18, Definition 3.2.4], see [*ibid.*, Remark 3.2.5]. It follows from the discussion in [*ibid.*, Remark 3.2.3] that we may identify the Mal'cev completion of $\Pi(\mathfrak{a}^+)$, denoted by $\Pi(\mathfrak{a}^+)_{\mathbb{Q}}$, with $\tilde{\Pi}(\mathfrak{a})$.

7.2.1. We write $N = \Pi(\mathfrak{a}^+)$ and we consider the Tate-linear variety $\mathrm{TL}(N)$ associated to N, defined as the fpqc quotient $N_{\mathbb{Q}}/N$. It follows from Lemma 4.3.12 that we may identify $\mathrm{TL}(N)$ with $Z(\mathfrak{a}^+)$. Given $\mathfrak{b} \subseteq \mathfrak{a}$ inducing $\mathfrak{b}^+ \subseteq \mathfrak{a}^+$, we get a map

$$\{\Pi_i(\mathfrak{b}^+), \mathrm{Fil}^\bullet \Pi_i(\mathfrak{b}^+)\}_{i\in\mathbb{Z}_{\geq 0}} \to \{\Pi_i(\mathfrak{a}^+), \mathrm{Fil}^\bullet \Pi_i(\mathfrak{a}^+)\}_{i\in\mathbb{Z}_{\geq 0}}.$$

It is straightforward to see that the first is a terraced Tate unipotent subgroup of the second, as defined in [*ibid.*,Definition 3.2.10]. If we write $N' = \Pi(\mathfrak{b}^+)$ and $N = \Pi(\mathfrak{a}^+)$, then we get an inclusion $N' \to N$ of Tate unipotent groups which is co-torsion free in the sense that $N' = N \cap N'_{\mathbb{Q}}$, see[*ibid.*,Lemma 3.2.11]. This induces an inclusion of formal Lie varieties

$$\mathrm{TL}(N') \subseteq \mathrm{TL}(N)$$

which in our notation is $Z(\mathfrak{b}^+) \subseteq Z(\mathfrak{a}^+)$.

7.2.2. If we consider $\mathbb{X}(\mathfrak{a}^+)$ equipped with its Lie bracket, then it is a Tate unipotent Lie $\mathbb{Z}_p$-algebra in the sense of [*ibid.*, Definition 3.2.13.(a)]. It follows that $\tilde{\mathbb{X}}(\mathfrak{a})$ is a Tate unipotent Lie $\mathbb{Q}_p$-algebra in the sense of [*ibid.*, Definition 3.2.13.(b)]. We may moreover identify $\tilde{\Pi}(\mathfrak{a})$ with the $\tilde{\mathbb{X}}(\mathfrak{a})$ equipped with the group structure coming from the BCH formula.

Proof of Theorem 7.1.1. Under the assumptions of Theorem 7.1.1, it follows from [*ibid.*, Theorem 5.1] that there is an inclusion $N' \subseteq \Pi(\mathfrak{a}^+)$ of Tate unipotent groups which is co-torsion free, such that $Z = \mathrm{TL}(N')$. This induces an inclusion $N'_{\mathbb{Q}} \subseteq \tilde{\Pi}(\mathfrak{a})$ of unipotent groups which corresponds to an inclusion of Tate unipotent Lie $\mathbb{Q}_p$-algebras

$$\mathfrak{Lie} N'_{\mathbb{Q}} \subseteq \tilde{\mathbb{X}}(\mathfrak{a}),$$

where $\mathfrak{Lie}$ denotes the $\mathbb{Q}_p$-Lie algebra sheaf associated to a unipotent group. By [*ibid.*, Lemma 3.2.19, Lemma 3.3.4], there is an F-stable Lie subalgebra $\mathfrak{b}_Z \subseteq \mathfrak{a}$ such that $\mathfrak{Lie} N'_{\mathbb{Q}} = \tilde{\mathbb{X}}(\mathfrak{b})$. This implies that $N'_{\mathbb{Q}} = \tilde{\Pi}(\mathfrak{b}_Z)$ and $N' = \Pi(\mathfrak{b}_Z^+)$, thus $Z = \mathrm{TL}(N') = Z(\mathfrak{b}_Z^+)$ as desired. □

8. Proof of the main theorem and some variants

8.1. **Preliminaries on Hecke operators.** Let (G, X) be a Shimura datum of Hodge type with reflex field E and let $p > 2$ be a prime such that $G = \mathsf{G}_{\mathbb{Q}_p}$ is quasi-split and split over an unramified extension. Let $U_p \subseteq G(\mathbb{Q}_p)$ be a hyperspecial subgroup and let $U^p \subseteq \mathsf{G}(\mathbb{A}_f^p)$ be a sufficiently small compact open subgroup. Let $\mathbf{Sh}_{G,U}$ be the Shimura variety of level $U = U^p U_p$ over E and for a prime $v|p$ of E let $E = \mathsf{E}_v$ and let $\mathscr{S}_{G,U}/\mathcal{O}_E$ be the canonical integral model of $\mathbf{Sh}_{G,U}$ constructed in [51].

Let $\mathrm{Sh}_{\mathsf{G},U}$ be the base change to $\overline{\mathbb{F}}_p$ of this integral canonical model for some choice of map $\mathcal{O}_{E,v} \to \overline{\mathbb{F}}_p$ and let

$$\mathrm{Sh}_{\mathsf{G},U_p} := \varprojlim_{K^p \subseteq \mathsf{G}(\mathbb{A}_f^p)} \mathrm{Sh}_{\mathsf{G},K^pU_p},$$

which is equipped with an action of $\mathsf{G}(\mathbb{A}_f^p)$. Note that the map

$$\pi : \mathrm{Sh}_{\mathsf{G},U_p} \to \mathrm{Sh}_{\mathsf{G},U}$$

is a pro-étale U^p-torsor.

Let $\mathsf{G}^{\mathrm{sc}} \to \mathsf{G}^{\mathrm{der}}$ be the simply-connected cover of the derived subgroup of G; we will often identify groups like $\mathsf{G}^{\mathrm{sc}}(\mathbb{A}_f^p)$ and $\mathsf{G}^{\mathrm{sc}}(\mathbb{Q}_\ell)$ with their images in $\mathsf{G}(\mathbb{A}_f^p)$ and $\mathsf{G}(\mathbb{Q}_\ell)$. Note that $\mathsf{G}^{\mathrm{sc}}(\mathbb{A}_f^p)$ acts on $\mathrm{Sh}_{\mathsf{G},U_p}$ via the natural map $\mathsf{G}^{\mathrm{sc}}(\mathbb{A}_f^p) \to \mathsf{G}(\mathbb{A}_f^p)$.

Let $Z \subseteq \mathrm{Sh}_{\mathsf{G},U}$ be a locally closed subvariety and let $\tilde{Z}$ be the inverse image of Z under π. We say that Z is *stable under the prime-to-p Hecke operators*, or that Z is $\mathsf{G}(\mathbb{A}_f^p)$*-stable*, if $\tilde{Z}$ is $\mathsf{G}(\mathbb{A}_f^p)$-stable. Similarly we say that Z is *stable under the reduced prime-to-p Hecke operators*, or that Z is $\mathsf{G}^{\mathrm{sc}}(\mathbb{A}_f^p)$*-stable*, if $\tilde{Z}$ is $\mathsf{G}^{\mathrm{sc}}(\mathbb{A}_f^p)$-stable. For $\ell \neq p$ we say that Z is $\mathsf{G}(\mathbb{Q}_\ell)$*-stable* if $\tilde{Z}$ is $\mathsf{G}(\mathbb{Q}_\ell)$-stable.

The *prime-to-p Hecke orbit* of a point $x \in \mathrm{Sh}_{\mathsf{G},U}(\overline{\mathbb{F}}_p)$ is defined to be the image in $\mathrm{Sh}_{\mathsf{G},U}(\overline{\mathbb{F}}_p)$ of $\mathsf{G}(\mathbb{A}_f^p) \cdot \tilde{x}$, for any choice of lift of $\tilde{x} \to \mathrm{Sh}_{\mathsf{G},U_p}(\overline{\mathbb{F}}_p)$. This does not depend on the choice of $\tilde{x}$ since it can be identified with the image in $\mathrm{Sh}_{\mathsf{G},U}(\overline{\mathbb{F}}_p)$ of the $\mathsf{G}(\mathbb{A}_f^p)$-orbit of $\pi^{-1}(x)$. We define the *reduced prime-to-p Hecke orbit* of a point $x \in \mathrm{Sh}_{\mathsf{G},U}(\overline{\mathbb{F}}_p)$ to be the image in $\mathrm{Sh}_{\mathsf{G},U}(\overline{\mathbb{F}}_p)$ of the $\mathsf{G}^{\mathrm{sc}}(\mathbb{A}_f^p)$ orbit of $\pi^{-1}(x)$. For $\ell \neq p$ we define the ℓ-adic Hecke orbit or $\mathsf{G}(\mathbb{Q}_\ell)$-Hecke orbit of a point x to be the image in $\mathrm{Sh}_{\mathsf{G},U}(\overline{\mathbb{F}}_p)$ of the $\mathsf{G}(\mathbb{Q}_\ell)$ orbit of $\pi^{-1}(x)$.

Remark 8.1.1. For $g \in \mathsf{G}(\mathbb{A}_f^p)$ and $U^p \subset \mathsf{G}(\mathbb{A}_f^p)$ there is a finite étale correspondence

$$\begin{array}{ccccc} & & \mathrm{Sh}_{\mathsf{G},U_p(U^p \cap gU^pg^{-1})} & & \\ & \swarrow^{p_1} & & \searrow^{p_2} & \\ \mathrm{Sh}_{\mathsf{G},U^pU_p} & & & & \mathrm{Sh}_{\mathsf{G},U_pgU^pg^{-1}} \xrightarrow{\;g\;} \mathrm{Sh}_{\mathsf{G},U^pU_p}\,. \end{array}$$

and the Hecke operator attached to g is $g \circ p_2 \circ p_1^{-1}$. A locally closed subvariety $Z \subseteq \mathrm{Sh}_{\mathsf{G},U^pU_p}$ is stable under the Hecke operator attached to g if and only if $\tilde{Z}$ is stable under the action of g considered as an element of $\mathsf{G}(\mathbb{A}_f^p)$.

8.2. **Local stabiliser principle.** Choose a Hodge embedding $(\mathsf{G}, \mathsf{X}) \to (\mathsf{G}_V, \mathsf{H}_V)$ as in Section 5.4. In particular, there is a self-dual $\mathbb{Z}_{(p)}$-lattice $V_{(p)}$ such that K_p is the stabiliser in $G(\mathbb{Q}_p)$ of $V_p := V_{(p)} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p$. Then for every sufficiently small compact open subgroup $U^p \subseteq \mathsf{G}(\mathbb{A}_f^p)$, we can find $\mathcal{U}^p \subseteq \mathsf{G}_V(\mathbb{A}_f^p)$ and a closed immersion

$$\mathrm{Sh}_{\mathsf{G},U} \hookrightarrow \mathrm{Sh}_{\mathsf{G}_V,\mathcal{U}}$$

Fix a point $x \in \mathrm{Sh}_{\mathsf{G},U}(\overline{\mathbb{F}}_p)$ such that $Y = A_x[p^\infty]$ is completely slope divisible. We write $[\![b]\!] := [\![b_x]\!]$ for the $\mathcal{G}(\breve{\mathbb{Z}}_p)$-$\sigma$-conjugacy class of elements of $G(\breve{\mathbb{Q}}_p)$ associated to x, and let $C_{\mathsf{G},[\![b]\!]} \subseteq \mathrm{Sh}_{\mathsf{G},U,[b]}$ be associated central leaf. Then we have seen that the profinite group $\mathbf{Aut}_G(Y)(\overline{\mathbb{F}}_p)$ acts on $\mathbf{Def}_{\mathrm{sus}}(Y)$.

For $x \in \mathrm{Sh}_{\mathsf{G},U}(\overline{\mathbb{F}}_p)$ we let I_x be the algebraic group over $\mathbb{Q}$ consisting of tensor preserving self-quasi-isogenies of the abelian variety A_x introduced in [52, Section 2.1.2]. By definition it is a closed subgroup of the algebraic group Aut_x over $\mathbb{Q}$, whose R-points are given by

$$\mathrm{Aut}_x(R) = \left(\mathrm{End}_{\overline{\mathbb{F}}_p}(A_x) \otimes_{\mathbb{Z}} R\right)^{\times}.$$

We let $I_x(\mathbb{Z}_{(p)}) \subseteq I_x(\mathbb{Q})$ be the intersection of $I_x(\mathbb{Q})$ with $\left(\mathrm{End}_{\overline{\mathbb{F}}_p}(A_x) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}\right)^{\times}$. Then for a lift $\tilde{x} \in \mathrm{Sh}_{\mathsf{G},U_p}(\overline{\mathbb{F}}_p)$ of x the stabiliser of $\tilde{x}$ in $\mathsf{G}(\mathbb{A}_f^p)$ is equal to $I_x(\mathbb{Z}_{(p)}) \subseteq I_x(\mathbb{Q}) \subseteq \mathsf{G}(\mathbb{A}_f^p)$, by [42, Lemma 6.1.3]. The following result is [42, Proposition 6.1.1], see also [15, Theorem 9.5] for the Siegel case. Recall that $C_{\mathsf{G},[\![b]\!]}^{/x}$ admits a closed immersion into $\mathbf{Def}_{\mathrm{sus}}(Y)$.

Proposition 8.2.1 (Local stabiliser principle)**.** *If $Z \subseteq C_{\mathsf{G},[\![b]\!]}$ is a $\mathsf{G}(\mathbb{A}_f^p)$-stable reduced closed subset containing x, then $Z^{/x} \subseteq C_{\mathsf{G},[\![b]\!]}^{/x}$ is stable under the action of $I_x(\mathbb{Z}_{(p)}) \subseteq \mathbf{Aut}_G(Y)(\overline{\mathbb{F}}_p)$ on $\mathbf{Def}_{\mathrm{sus}}(Y)$.*

Remark 8.2.2. The same proof shows that for any $\mathsf{G}^{\mathrm{sc}}(\mathbb{A}_f^p)$-stable reduced closed subset $Z \subseteq C_{\mathsf{G},[\![b]\!]}$ containing x, the subscheme $Z^{/x} \subseteq C_{\mathsf{G},[\![b]\!]}^{/x}$ is stable under the action of $I_x(\mathbb{Z}_{(p)}) \cap \mathsf{G}^{\mathrm{sc}}(\mathbb{A}_f^p)$ on $\mathbf{Def}_{\mathrm{sus}}(Y)$.

8.3. **Proof of Theorem I.** We keep the notation as in Section 8.1 and Section 8.2. Let $\mathsf{G}^{\mathrm{ad}} = \mathsf{G}_1 \times \cdots \times \mathsf{G}_n$ be a decomposition of G^{ad} into a product of $\mathbb{Q}$-simple groups and write $G_i = \mathsf{G}_i \otimes \mathbb{Q}_p$ for $i = 1, \cdots, n$. Recall that for a reductive group $\mathbb{G}$ over $\mathbb{Q}_p$ we denote by $B(\mathbb{G})$ the set of σ-conjugacy classes in $\mathbb{G}(\breve{\mathbb{Q}}_p)$.

Definition 8.3.1 (Definition 5.3.2 of [55])**.** An element $[b] \in B(G_{\mathbb{Q}_p})$ is called $\mathbb{Q}$-*non-basic* if the image $[b_i]$ of $[b]$ in $B(G_{i,\mathbb{Q}_p})$ is non-basic for all i.

Let $C_{\mathsf{G},[\![b]\!]} \subseteq \mathrm{Sh}_{\mathsf{G},U,[b]}$ be a central leaf defined as in Section 5.4 and let ν_b be the Newton cocharacter of b for some $b \in [b]$, see [54, Section 1.1.2] for the definition of the Newton cocharacter. Let $P_{\nu_b} \subseteq G_{\breve{\mathbb{Q}}_p}$ be the associated parabolic subgroup with unipotent radical U_{ν_b}.

Theorem 8.3.2. *Let $Z \subseteq C_{\mathsf{G},[\![b]\!]}$ be a $\mathsf{G}(\mathbb{A}_f^p)$-stable closed subvariety. If b is $\mathbb{Q}$-non-basic, then $Z = C_{\mathsf{G},[\![b]\!]}$.*

Theorem 8.3.2 clearly implies Theorem I. The discussion in [55, Section 1.6] implies that the theorem is also true when $[b]$ is $\mathbb{Q}$-basic, that is, when $[b_i]$ is basic for all i. In this case the central leaves are finite and the claim is that $\mathsf{G}(\mathbb{A}_f^p)$ acts transitively on them. We expect Theorem 8.3.2 to be true for arbitrary $[b]$, but we don't know how to prove the discrete part.

Proof. We reduce immediately to the case that Z is the Zariski closure of the $\mathsf{G}(\mathbb{A}_f^p)$-orbit of a point. It follows from [42, Lemma 3.1.2] that such a Z is itself $\mathsf{G}(\mathbb{A}_f^p)$-stable.

By [55, Theorem C], which states that under our assumptions a $\mathsf{G}(\mathbb{A}_f^p)$-stable subvariety $Z \subseteq C_{\mathsf{G},[\![b]\!]}$ intersects each connected component of $C_{\mathsf{G},[\![b]\!]}$ non-trivially, it suffices to show that Z is equidimensional of the same dimension as $C_{\mathsf{G},[\![b]\!]}$.

It follows from [47, Proposition 2.4.5] that there is a central leaf $C_{\mathsf{G},[\![b]\!]} \subseteq \mathrm{Sh}_{\mathsf{G},U,[b]}$ such that the universal p-divisible group over $C_{\mathsf{G},[\![b]\!]}$ is completely slope divisible. Since $C_{\mathsf{G},[\![b]\!]}$ and $C_{\mathsf{G},[\![b]\!]}$ share a $\mathsf{G}(\mathbb{A}_f^p)$-equivariant finite étale cover, it suffices to this equidimensionality for $C_{\mathsf{G},[\![b']\!]}$ and therefore we will assume without loss of generality that the universal p-divisible group over $C_{\mathsf{G},[\![b]\!]}$ is completely slope divisible. By [42, Lemma 3.1.1], the smooth locus Z^{sm} of Z is also $\mathsf{G}(\mathbb{A}_f^p)$-stable. It is moreover explained in [42, Section 3.3] that the abelian variety up to prime-to-p isogeny over $\mathrm{Sh}_{\mathsf{G}_V,\mathcal{U}}$ induces an (overconvergent) F-isocrystal $\mathcal{M}$ over $\mathrm{Sh}_{\mathsf{G},U}$.

The assumption that $[b]$ is $\mathbb{Q}$-non-basic allows us to invoke [42, Corollary 3.3.5][11], which tells us that the unipotent radical of the monodromy group of $\mathcal{M}$ over Z^{sm} is isomorphic the unipotent radical of P_{ν_b}. Theorem II then tells us that for $x \in Z^{\mathrm{sm}}(\overline{\mathbb{F}}_p)$ the monodromy of the isocrystal $\mathcal{M}$ over $\operatorname{Spec} \widehat{\mathcal{O}}_{Z,x}$ is equal to U_{ν_b}.

The assumption that the universal p-divisible group over $C_{\mathsf{G},[\![b]\!]}$ is completely slope divisible tells us that $Z^{/x} \subseteq \mathbf{Def}_{\mathrm{sus}}(Y)$ for $Y = A_x[p^\infty]$. Theorem 6.1.1 tells us that $Z^{/x}$ is not contained in $Z(\mathfrak{b}^+)$ for any F-stable Lie algebra $\mathfrak{b} \subsetneq \mathfrak{a} = \mathbb{D}(\mathcal{H}_Y^{G,\circ})[\frac{1}{p}] = \operatorname{Lie} U_{\nu_b}$.

Proposition 8.2.1 tells us that $Z^{/x}$ is stable under the action of

$$I'_x(\mathbb{Z}_{(p)}) \subseteq \mathbf{Aut}_G(Y)(\overline{\mathbb{F}}_p).$$

By continuity it is also stable under its closure $\Gamma \subseteq \mathbf{Aut}_G(Y)(\overline{\mathbb{F}}_p)$. It follows as in the proof of [42, Corollary 6.1.6] that Γ acts strongly non-trivially on $C_{\mathsf{G},[\![b]\!]}^{/x} = Z(\mathfrak{a}^+)$. Therefore Theorem 7.1.1 tells us that $Z^{/x} = Z(\mathfrak{b}^+)$ for some F-stable Lie algebra $\mathfrak{b} \subseteq \mathfrak{a}$[12], and the previous paragraph tells us that $\mathfrak{a} = \mathfrak{b}$. In other words, $Z^{/x} = C_{\mathsf{G},[\![b]\!]}^{/x}$ for all points $x \in Z^{\mathrm{sm}}(\overline{\mathbb{F}}_p)$. Since $Z^{\mathrm{sm}} \subseteq Z$ is dense because Z is reduced, it follows that Z is equidimensional of the same dimension as $C_{\mathsf{G},[\![b]\!]}$, and therefore we are done. □

8.4. **Isogeny classes are dense in Newton strata.** Let (G, X) be a Shimura variety of Hodge type, let $x \in \mathrm{Sh}_{\mathsf{G},U,[b]}(\overline{\mathbb{F}}_p)$ and let $\mathscr{I}_x \subseteq \mathrm{Sh}_{\mathsf{G},U,[b]}(\overline{\mathbb{F}}_p)$ be the isogeny class of x in the sense of [52].

Theorem 8.4.1. *If $W \subseteq \mathrm{Sh}_{\mathsf{G},U,[b]}$ is a $\mathsf{G}(\mathbb{A}_f^p)$ closed subset of a $\mathbb{Q}$-non-basic Newton stratum $\mathrm{Sh}_{\mathsf{G},U,[b]} \subseteq \mathrm{Sh}_{\mathsf{G},U}$, then $W = \mathrm{Sh}_{\mathsf{G},U,[b]}$.*

Proof. The isogeny class $\mathscr{I}_x \subseteq \mathrm{Sh}_{\mathsf{G},U,[b]}(\overline{\mathbb{F}}_p)$ intersects every central leaf $C_{\mathsf{G},[\![b']\!]} \subseteq \mathrm{Sh}_{\mathsf{G},U,[b]}$ by the Rapoport–Zink uniformisation of isogeny classes (which follows from the main result of [52], see [*ibid*, Section 1.4]). Thus if $W \subseteq \mathrm{Sh}_{\mathsf{G},U,[b]}$ is a $\mathsf{G}(\mathbb{A}_f^p)$-stable closed subset containing $\mathscr{I}_x \subseteq \mathrm{Sh}_{\mathsf{G},U,[b]}(\overline{\mathbb{F}}_p)$, then W intersects every central leaf $C_{\mathsf{G},[\![b']\!]} \subseteq \mathrm{Sh}_{\mathsf{G},U,[b]}$ in a $\mathsf{G}(\mathbb{A}_f^p)$-stable *non-empty* closed subset W_C. Theorem 8.3.2 now tells us that $W_C = C$ and since $\mathrm{Sh}_{\mathsf{G},U,[b]}$ is the (set-theoretic) union of all the central leaves it follows that $W = \mathrm{Sh}_{\mathsf{G},U,[b]}$. □

8.5. **Orthogonal Shimura varieties.** A conjecture of Bragg–Yang, see [6, Conjecture 8.2], predicts that prime-to-p Hecke orbits are Zariski dense in certain Newton strata of certain *orthogonal* Shimura varieties. These are the Shimura varieties for the group $\mathrm{SO}(M)$ where M is a quadratic space over $\mathbb{Q}$ with signature $(2, m-2)$; they are of abelian type by [61, Appendix B].

In general one does *not* expect that prime-to-p Hecke orbits are Zariski dense in Newton strata, but when the Shimura datum is fully Hodge–Newton decomposable at p, then $\mathbb{Q}$-non-basic Newton strata are equal to central leaves by [74, Theorem E.(2)]. By [36, Theorem D], the orthogonal Shimura varieties in question are indeed fully Hodge–Newton decomposable at p.

[11]The statement of [42, Corollary 3.3.5] contains the assumption that [42, Hypothesis 2.3.1] holds. This is true for us because K_p is hyperspecial, see [42, Lemma 2.3.2].

[12]To be precise, Theorem 7.1.1 states that the inverse image of $Z^{/x}$ in $Z(p^2\mathfrak{a}^+)$ is of the form $Z(p^2\mathfrak{b}^+)$ for some F-stable Lie algebra $\mathfrak{b} \subseteq \mathfrak{a}$; this implies that $Z = Z(\mathfrak{b}^+)$.

It follows from the results of [55] and the proof of [43, Theorem 6.0.7] that

$$\pi_0(\mathrm{Sh}_{\mathsf{G},U,[b]}) \to \pi_0(\mathrm{Sh}_{\mathsf{G},U})$$

is a bijection for $\mathbb{Q}$-non-basic b for Shimura varieties of Hodge type. Since Newton strata behave well with respect to the dévissage from Hodge type to abelian type, this result also holds for Shimura varieties of abelian type (see [75, Section 5.5]). Thus [6, Conjecture 8.2] comes down to showing that the Zariski closure of prime-to-p Hecke orbits have the correct dimensions, and this can be reduced to the Hodge type case and then to Theorem 8.3.2 as in the proof of [42, Corollary 6.4.1].

Remark 8.5.1. This line of reasoning shows more generally that the Hecke orbit conjecture holds for fully Hodge–Newton decomposable Shimura varieties of abelian type, at primes $p > 2$ of hyper-special good reduction, for central leaves in Newton strata corresponding to $\mathbb{Q}$-non-basic $[b]$.

8.6. **ℓ-power Hecke orbits.** In this section we study the Zariski closures of ℓ-adic Hecke orbits of points for primes $\ell \neq p$. Since the ℓ-adic Hecke operators do not, generally, act transitively on $\pi_0(\mathrm{Sh}_{\mathsf{G},U})$, all we can hope to prove is that ℓ-adic Hecke orbits are dense in a union of connected components of a central leaf. However, this cannot be true if $G_{\mathbb{Q}_\ell}$ is totally anisotropic because then there aren't enough ℓ-power Hecke operators. Let $C_{\mathsf{G},[\![b]\!]} \subseteq \mathrm{Sh}_{\mathsf{G},U,[b]}$ be a central leaf as before and assume that $[b]$ is $\mathbb{Q}$-non-basic.

Theorem 8.6.1. *If $\mathrm{Sh}_{\mathsf{G},U}$ is proper and $G_{\mathbb{Q}_\ell}$ is totally isotropic, then any $\mathsf{G}(\mathbb{Q}_\ell)$-stable reduced closed subscheme $Z \subseteq C_{\mathsf{G},[\![b]\!]}$ is a union of connected components of C.*

We start by proving a lemma, cf. [84, Lemma 3.3.2].

Lemma 8.6.2. *Let ℓ be a prime such that $G_{\mathbb{Q}_\ell}$ is totally isotropic. If $Z \subseteq \mathrm{Sh}_{\mathsf{G},U}$ is a finite scheme that is $\mathsf{G}(\mathbb{Q}_\ell)$-stable, then Z is contained in the basic locus of $\mathrm{Sh}_{\mathsf{G},U}$.*

Proof. Let $\tilde{x} \in \mathrm{Sh}_{\mathsf{G},U_p}(\overline{\mathbb{F}}_p)$ with image $x \in Z(\overline{\mathbb{F}}_p)$. Let $I_x(\mathbb{Z}_{(p)}) \subseteq \mathsf{G}(\mathbb{A}_f^p)$ be the group of tensor-preserving automorphisms as in Section 8.2. Let U_ℓ be the image of U^p under $\mathsf{G}(\mathbb{A}_f^p) \to \mathsf{G}(\mathbb{Q}_\ell)$ and identify $I_x(\mathbb{Z}_{(p)})$ with its image under $\mathsf{G}(\mathbb{A}_f^p) \to \mathsf{G}(\mathbb{Q}_\ell)$. Then the ℓ-adic Hecke orbit of x can be written as

$$I_x(\mathbb{Z}_{(p)})\backslash \mathsf{G}(\mathbb{Q}_\ell)/U_\ell,$$

which is finite by assumption. Since the closure of $I_x(\mathbb{Z}_{(p)})$ has finite index in $I_x(\mathbb{Q}_\ell)$, it follows that

$$I_x(\mathbb{Q}_\ell)\backslash \mathsf{G}(\mathbb{Q}_\ell)$$

is compact. Since $I_{x,\mathbb{Q}_\ell}$ is connected it follows from of [5, Propositions 8.4, Proposition 9.3] that it is a maximal trigonalizable subgroup of $\mathsf{G}_{\mathbb{Q}_\ell}$. Since $\mathsf{G}_{\mathbb{Q}_\ell}$ is totally anisotropic, $I_{x,\mathbb{Q}_\ell}$ must be a parabolic subgroup of $\mathsf{G}_{\mathbb{Q}_\ell}$, and because it is reductive it follows that $I_{x,\mathbb{Q}_\ell} = \mathsf{G}_{\mathbb{Q}_\ell}$. It is well known that this only happens when x is contained in the basic locus. □

Lemma 8.6.3. *Assume that $\mathrm{Sh}_{\mathsf{G},U}$ is proper. If a reduced closed subscheme $Z \subseteq \mathrm{Sh}_{\mathsf{G},U,[b]}$ is stable under the action of $\mathsf{G}(\mathbb{Q}_\ell)$ for some $\ell \neq p$ such that $\mathsf{G}^{\mathrm{sc}} \otimes \mathbb{Q}_\ell$ is totally isotropic, then Z is stable under the action of $\mathsf{G}^{\mathrm{sc}}(\mathbb{A}_f^p)$.*

Proof. The proof is almost exactly the same as the proof of [11, Proposition 4.6]. Nevertheless, we will give a complete proof for the benefit of the reader.

Step 1. A standard argument (see e.g. [84, The proof of Proposition 3.3.1]) using the quasi-affineness of the Ekedahl–Oort stratification ([35, Corollary I.2.6]) and the properness of $\mathrm{Sh}_{\mathsf{G},U}$ shows that the Zariski closure $\overline{Z}$ of Z in $\mathrm{Sh}_{\mathsf{G},U}$ contains a point $x \in \mathrm{Sh}_{\mathsf{G},U}(\overline{\mathbb{F}}_p)$ with finite ℓ-power Hecke orbit. Let $Z' \subseteq Z$ be the union of irreducible components V of Z whose closure $\overline{V}$ does not contain a point with a finite ℓ-power orbit. Suppose for the sake of contradiction that Z' is non-empty, then Z' is a $\mathsf{G}(\mathbb{Q}_\ell)$-stable reduced locally closed subscheme of $\mathrm{Sh}_{\mathsf{G},U}$ and thus $\overline{Z'}$ contains a point with a finite ℓ-power orbit, which is a contradiction.

It follows from Lemma 8.6.2 that x is contained in the basic locus of $\mathrm{Sh}_{\mathsf{G},U}$, and moreover that I_x is an inner form of G (see [39, Corollary 5.2.11]). Thus the isomorphism $I_x \otimes \mathbb{A}_f^p \to \mathsf{G} \otimes \mathbb{A}_f^p$ induces an isomorphism $I_x^{\mathrm{sc}} \otimes \mathbb{A}_f^p \to \mathsf{G}^{\mathrm{sc}} \otimes \mathbb{A}_f^p$. Strong approximation away from ∞, ℓ (see [68, Theorem 7.8]), using the fact that $\mathsf{G}_{\mathbb{Q}_\ell}$ is totally isotropic, tells us that the image of

$$I_x^{\mathrm{sc}}(\mathbb{Z}_{(p)}) \to \mathsf{G}^{\mathrm{sc}}(\mathbb{A}_f^{p,\ell})$$

is dense. Since the inclusion of the ℓ-adic Hecke orbit of x inside the prime-to-p Hecke orbit of x can be identified with

$$I_x(\mathbb{Z}_{(p)})\backslash \mathsf{G}(\mathbb{Q}_\ell)U^p/U^p \subseteq I_x(\mathbb{Z}_{(p)})\backslash \mathsf{G}(\mathbb{A}_f^p)/U^p,$$

we see that the ℓ-adic Hecke orbit of x is $\mathsf{G}^{\mathrm{sc}}(\mathbb{A}_f^p)$-stable. Moreover the local stabiliser principle, see Remark 8.2.2, tells us that

$$\overline{Z}^{/x} \subseteq C_{\mathsf{G},[\![b]\!]}^{/x}$$

is kept stable under the action of

$$I_x(\mathbb{Z}_{(p)}) \cap \mathsf{G}^{\mathrm{sc}}(\mathbb{A}_f^p) \subseteq \mathbf{Aut}_G(Y)(\overline{\mathbb{F}}_p)$$

and by continuity it is also kept stable by its closure.

Step 2. Take a λ-adic Hecke operator $g_\lambda \in \mathsf{G}^{\mathrm{sc}}(\mathbb{Q}_\lambda)$ for $\lambda \neq p, \ell$ and let W be the image of $\overline{Z}$ under g_λ. Since every irreducible component of $\overline{Z}$ contains a point x with finite ℓ-power Hecke orbit, it follows that every irreducible component of W contains an element in the g_λ-orbit of such an x. Since $\overline{Z}$ contains the $\mathsf{G}^{\mathrm{sc}}(\mathbb{A}_f^p)$ orbit of x, it follows that every irreducible component W_i of W intersects $\overline{Z}$ in a point y_i with finite ℓ-power Hecke orbit.

The reduced prime-to-p Hecke orbit of y_i has the form

$$I_{y_i}^{\mathrm{sc}}(\mathbb{Z}_{(p)})\backslash \mathsf{G}^{\mathrm{sc}}(\mathbb{A}_f^p)U^p/U^p.$$

By strong approximation ([68, Theorem 7.8]) away from ℓ for $I_{y_i}^{\mathrm{sc}}$, using the fact that $\mathsf{G}_{\mathbb{Q}_\ell}$ is totally isotropic, we can choose $\delta \in I_{y_i}^{\mathrm{sc}}(\mathbb{Q})$ which lands in $U^{p,\ell,\lambda}$ and such that there is an element $g_\ell \in \mathsf{G}^{\mathrm{sc}}(\mathbb{Q}_\ell)$ such that $\delta \cdot g_\ell = g_\lambda^{-1}$ in g_λ in $I_{y_i}(\mathbb{Z}_{(p)})\backslash \mathsf{G}^{\mathrm{sc}}(\mathbb{A}_f^p)U^p/U^p$.

We see that $g_\lambda \circ g_\ell$ fixes y_i, and since $\overline{Z}$ is g_ℓ-stable, the image of $\overline{Z}$ under $g_\lambda \circ g_\ell$ is equal to W. Now we consider the closed subschemes

$$W_i^{/y_i}, \overline{Z}^{/y_i} \subseteq \mathrm{Sh}_{\mathsf{G},U}^{/y_i}.$$

The subscheme $W_i^{/y_i}$ is the image of $\overline{Z}^{/y_i}$ under the Hecke operator $g_\lambda \circ g_\ell$. Since $\overline{Z}^{/y_i}$ is stable under the action of the closure of $I_x(\mathbb{Z}_{(p)})$ and since $g_\lambda \circ g_\ell$ is contained in that closure by construction, it follows that

$$W_i^{/y_i} \subseteq \overline{Z}^{/y_i}.$$

From this we deduce that $W_i \subseteq \overline{Z}$. Thus every irreducible component W_i of W is contained in $\overline{Z}$, and we conclude that $\overline{Z}$ is stable under the action of g_λ. Since λ and g_λ were arbitrary, it follows that $\overline{Z}$ is stable under the action of $\mathsf{G}^{\mathrm{sc}}(\mathbb{A}_f^p)$.

We know that Z is the intersection of $\overline{Z}$ with $\mathrm{Sh}_{\mathsf{G},U,[b]}$ and as the intersection of two $\mathsf{G}^{\mathrm{sc}}(\mathbb{A}_f^p)$-stable subschemes it must itself be $\mathsf{G}^{\mathrm{sc}}(\mathbb{A}_f^p)$-stable. □

Proof of Theorem 8.6.1. If we let $\mathcal{M}$ be the isocrystal attached to the universal abelian variety up to prime-to-p isogeny over Z and if we let $x \in Z$ be a smooth point, then arguing as in the proof of Theorem 8.3.2, we can combine [42, Corollary 3.3.5] with Theorem II to deduce that the monodromy of the isocrystal $\mathcal{M}$ over $\operatorname{Spec} \widehat{\mathcal{O}}_{Z,x}$ is isomorphic to U_{ν_b}.

Proposition 8.2.1 (see Remark 8.2.2) tells us that $Z^{/x}$ is stable under the action of

$$\left(I_x(\mathbb{Z}_{(p)}) \cap \mathsf{G}^{\mathrm{sc}}(\mathbb{A}_f^p)\right) \subseteq \mathbf{Aut}_G(Y)(\overline{\mathbb{F}}_p).$$

By continuity, it is also stable under the closure $\Gamma \subseteq \mathbf{Aut}_G(Y)(\overline{\mathbb{F}}_p)$. As in the proof of [42, Corollary 6.1.6], it follows that Γ acts strongly non-trivially on $C^{/x}_{\mathsf{G},[\![b]\!]} = Z(\mathfrak{a}^+)$. The same argument as in the proof of Theorem 8.3.2 allows us to conclude that Z is a union of connected components of C. □

8.7. **Further questions of Chai–Oort.** Let $Z \subseteq C_{(Y,\lambda)}$ be an irreducible smooth closed subvariety and let $x \in Z(\overline{\mathbb{F}}_p)$. We call Z *strongly Tate-linear at* x if $Z^{/x} \subseteq C^{/x}_{(Y,\lambda)}$ is a strongly Tate-linear subvariety.

Question 8.7.1. Suppose that Z is strongly Tate-linear at some closed point $x_0 \in Z(\overline{\mathbb{F}}_p)$. Is Z then strongly Tate-linear at all closed points $x \in Z(\overline{\mathbb{F}}_p)$?

It follows from Theorem II that the monodromy group of $\mathcal{M}$ over $\operatorname{Spec} \widehat{\mathcal{O}}_{Z,x}$ does not depend on x. Now the validity of Conjecture 5.5.5 would imply that the question above has an affirmative answer.

Question 8.7.2. Suppose that Z is strongly Tate-linear at some closed point $x_0 \in Z(\overline{\mathbb{F}}_p)$. Is Z an irreducible component of a central leaf in the mod p reduction of a Shimura variety of Hodge type?

The stronger assertion that Z must itself be an irreducible component of a Shimura variety of Hodge type is false in general, because only finitely many central leaves in a given Newton stratum contain the mod p reductions of special points by [53, Theorem 1.3]. If $C_{(Y,\lambda)}$ is the ordinary locus and Z is proper, then results towards this stronger assertion are proved in work of Moonen, [62].

8.8. **Results at ramified primes and parahoric level.** The statements of our theorems make sense for central leaves in special fibres of the Kisin–Pappas [50] integral models of Shimura varieties of parahoric level at tamely ramified primes $p > 2$.

8.8.1. If G is unramified over $\mathbb{Q}_p$ and the level is parahoric, then the Hecke orbit conjecture for central leaves at parahoric level follows immediately from Theorem 8.3.2. The main observation is that the forgetful map

$$C' \to C,$$

where C' is a central leaf at Iwahori level and C is a central leaf at hyperspecial level, is equivariant for the prime-to-p Hecke operators and induces a bijection on π_0. This last statement can be proven using the surjectivity of $C' \to C$[13], and the explicit description of connected components of Igusa varieties in [43, 55].

[13] This surjectivity is axiom 4c of the He-Rapoport axioms, see [74], and follows from Rapoport–Zink uniformisation at parahoric level, which is [41, Theorem 2].

Rapoport–Zink uniformisation of isogeny classes at parahoric level[13], implies as before that isogeny classes are dense in the Newton strata containing them.

8.8.2. If G is a ramified group over $\mathbb{Q}_p$, then it is not always true that $\mathsf{G}(\mathbb{A}_f^p)$ acts transitively on $\pi_0(\mathrm{Sh}_{\mathsf{G},U})$ (see [64] for explicit counterexamples), and therefore it is not necessarily true that $\mathsf{G}(\mathbb{A}_f^p)$ acts transitively on $\pi_0(C)$ either because $\pi_0(C) \to \pi_0(\mathrm{Sh}_{\mathsf{G},U})$ is surjective. Nevertheless, we expect that the continuous part of the Hecke orbit conjecture is true for ramified groups. In fact, we suspect that the strategy adopted in this paper can be made to work for ramified groups.

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